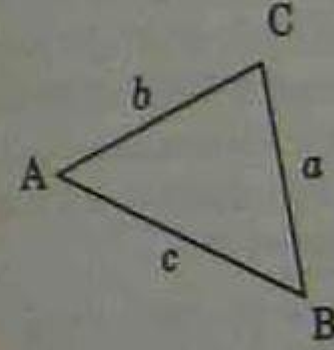
10.5 Triangles without right angles

Trigonometry can be used in all triangles regardless of whether they contain a right angle. There are two methods and each requires a formula. **THESE FORMULAE MUST BE KNOWN.**

10.5.1 The Sine Rule (Rule of opposites)

Use the Sine Rule when you know two angles and one side, or when you know two sides and an angle.

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$$

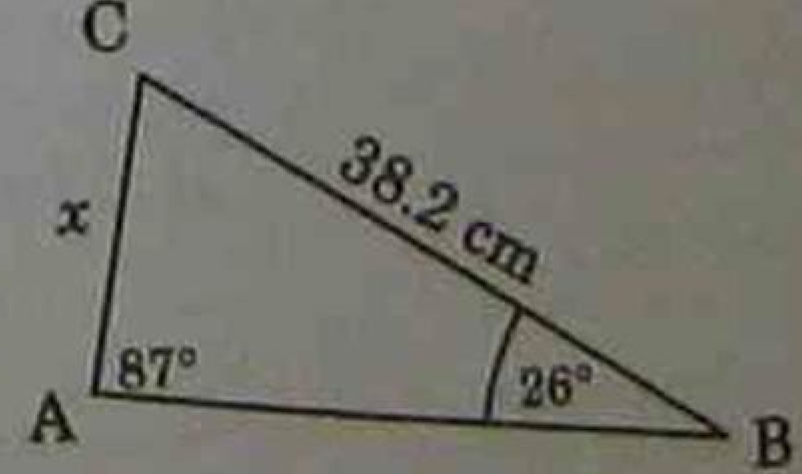


Note the opposites in the formula:

$$\frac{\text{side}}{\sin\,[\text{opposite angle}]} = \frac{\text{side}}{\sin\,[\text{opposite angle}]}$$

Examples

(a) Find x, correct to three significant figures.



SOLUTION

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

We have $\hat{A} = 87°$, $\hat{B} = 26°$ and $a = 38.2$.

We want b.

$$\frac{38.2}{\sin 87°} = \frac{x}{\sin 26°}$$

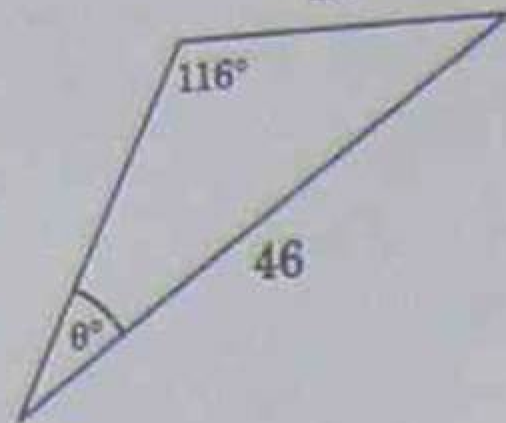
Therefore $$x = \frac{38.2 \sin 26°}{\sin 87°}$$
$$= 16.768\,759$$
$$= 16.8$$

(three significant figures)

Calculator

38.2 $\boxed{\times}$ 26 $\boxed{\text{SIN}}$ $\boxed{\div}$ 87 $\boxed{\text{SIN}}$ $\boxed{=}$

(b) Find θ to the nearest minute.



SOLUTION

$$\frac{25}{\sin\theta} = \frac{46}{\sin 116^\circ} \qquad \frac{\text{Side}}{\sin\text{(Opp)}}$$

$$\therefore 46\sin\theta = 25\sin 116^\circ$$

$$\sin\theta = \frac{25\sin 116^\circ}{46}$$

$$\therefore \quad \theta = 29^\circ 14'$$

Calculator

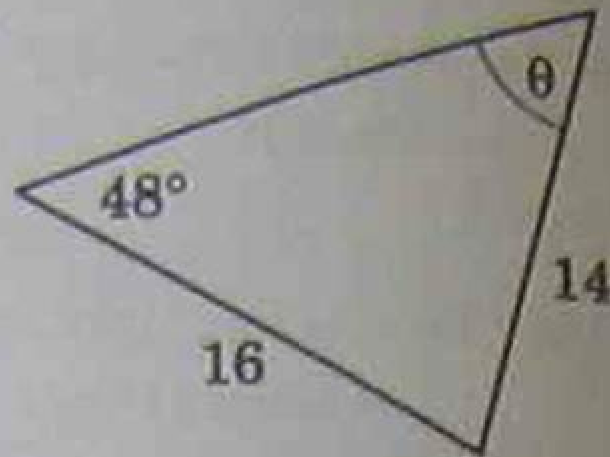
25 $\boxed{\times}$ 116 $\boxed{\text{SIN}}$ $\boxed{\div}$ 46 $\boxed{=}$

$\boxed{\text{INV}}$ $\boxed{\text{SIN}}$ $\boxed{\text{INV}}$ $\boxed{{}^\circ\ {}'\ {}''}$

The ambiguous case

Note: The required angle could also be $(180° - 29° 14')$, that is, $150° 46'$, but this would mean the angle sum of the triangle is greater than $180°$. As both acute and obtuse angles have sine positive, an obtuse answer is always possible. When a decision cannot be made between the acute result and the obtuse result it is known as the ambiguous case.

(c) Find θ to the nearest minute.



SOLUTION

$$\frac{16}{\sin\theta} = \frac{14}{\sin 48^\circ}$$

$$14\sin\theta = 16\sin 48^\circ$$

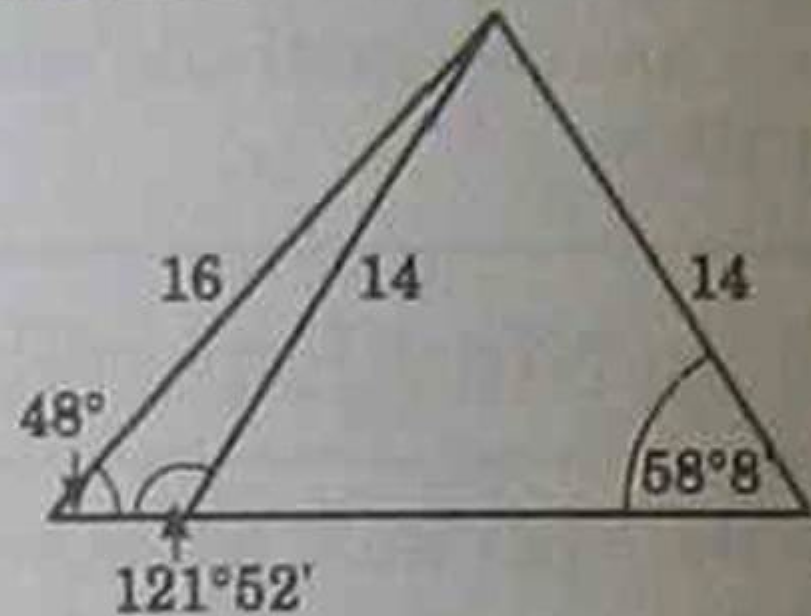
$$\sin\theta = \frac{16\sin 48^\circ}{14}$$

$$\theta = 58^\circ\, 8'$$

But also, $\theta = (180^\circ - 58^\circ\, 8')$

$$= 121^\circ\, 52'$$

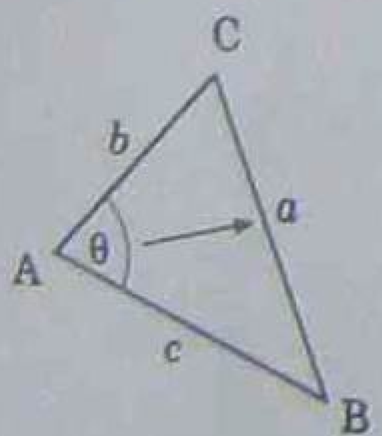
Either answer is possible. From this information,



This may happen when using the Sine Rule to find an angle. Check with the original information — the obtuse value may not be possible. Remember that the angle sum of the triangle is 180°.

10.5.2 The Cosine Rule

Finding a side



The Cosine Rule is used when the question involves two sides and the included angle.

$$\boxed{a^2 = b^2 + c^2 - 2bc \cos A}$$

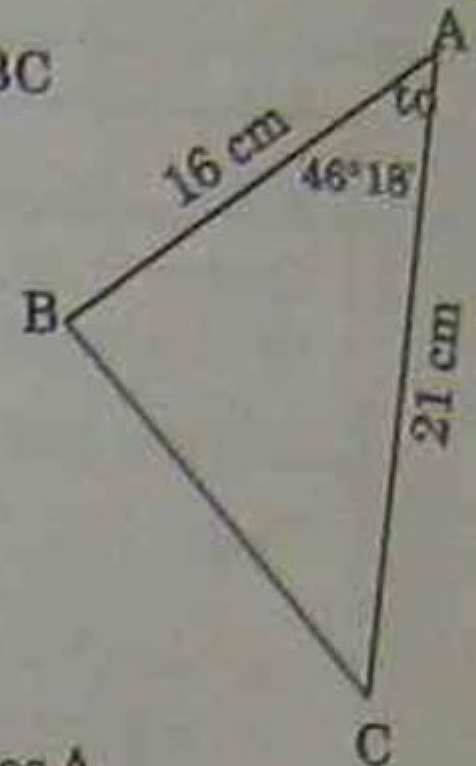
Sides around angle A

Angle opposite the side being found

Similar formulae can be written for b^2 and c^2.

Example

(a) Find the length of BC to the nearest cm.



SOLUTION

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$= 21^2 + 16^2 - 2 \times 21 \times 16 \times \cos 46°\,18'$$
$$= 232.727\,02$$

← Keep this on your screen.

$$a = \sqrt{232.727\,02}$$
$$= 15.255\,393$$
$$\approx 15$$

BC is 15 cm.

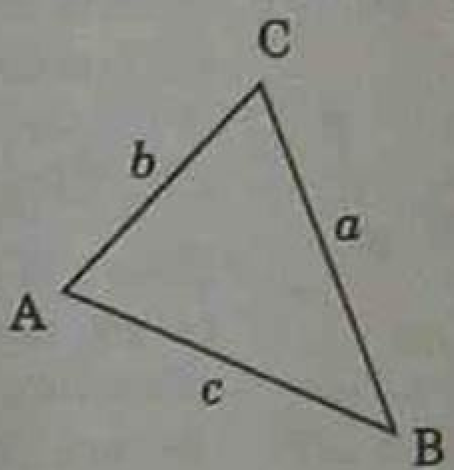
The complete working can be done on your calculator.

21 $\boxed{\text{INV}}$ $\boxed{\sqrt{\ }}$ (x^2) $\boxed{+}$ 16 $\boxed{\text{INV}}$ $\boxed{\sqrt{\ }}$ (x^2) $\boxed{-}$
2 $\boxed{\times}$ 21 $\boxed{\times}$ 16 $\boxed{\times}$ 46 $\boxed{\circ\,'\,''}$ 18 $\boxed{\circ\,'\,''}$
$\boxed{\text{COS}}$ $\boxed{=}$

If your calculator has a separate $\boxed{x^2}$ button, ignore the two $\boxed{\text{INV}}$.

This calculation gives a^2, so we need to push $\boxed{\sqrt{\ }}$.

To find an angle



The question must contain values for all three sides to be able to use the angle form of the Cosine Rule.

$$\boxed{\cos A = \frac{b^2 + c^2 - a^2}{2bc}}$$

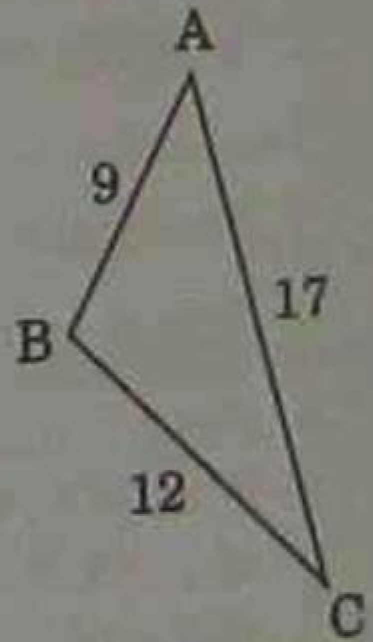
Similar forms can be written for cos B and cos C. Note that the side subtracted is the one opposite the required angle.

Example

(b) In the ΔABC, $a = 12$ cm, $b = 17$ cm and $c = 9$ cm. Find the size of the largest angle.

SOLUTION

The largest angle is always opposite the longest side. Therefore the angle required is B.



$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$= \frac{12^2 + 9^2 - 17^2}{2 \times 12 \times 9}$$

$$= \frac{\left(12^2 + 9^2 - 17^2\right)}{(2 \times 12 \times 9)}$$

Use brackets around top and bottom.

$= -0.296\ 2963$ ← Leave this on the screen.

$\therefore\ B = 107^\circ\ 14'$.

Calculator

$\boxed{(}$ 12 $\boxed{\text{INV}}$ $\boxed{\sqrt{\ }}$ (x^2) $\boxed{+}$ 9 $\boxed{\text{INV}}$ $\boxed{\sqrt{\ }}$ (x^2) $\boxed{-}$

17 $\boxed{\text{INV}}$ $\boxed{\sqrt{\ }}$ (x^2) $\boxed{)}$ $\boxed{\div}$ $\boxed{(}$ 2 $\boxed{\times}$ 12 $\boxed{\times}$

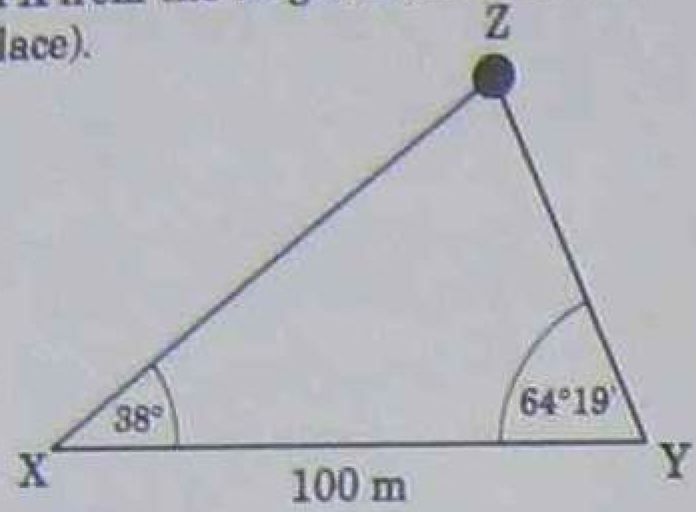
9 $\boxed{)}$ $\boxed{=}$

This gives cos B, then use

$\boxed{\text{INV}}$ $\boxed{\text{COS}}$ $\boxed{\text{INV}}$ $\boxed{{}^\circ\ '\ ''}$

10.5.3 Miscellaneous worked examples

(a) Two cannons X and Y, 100 metres apart, fire at a target Z. If $\angle ZXY = 38°$ and $\angle ZYX = 64° \, 19'$, find the distance of X from the target Z (one decimal place).



SOLUTION

The question asks us to find the length of XZ. We have two angles and a side — use the Sine Rule. *But*, the Sine Rule needs the angles opposite the sides. We must calculate the size of $\hat{Z}$.

of Z.

$\hat{Z} = 180° - (64° 19' + 38°)$ $\boxed{\text{angle sum of a } \Delta}$

$= 77° 41'$.

Then $\boxed{\dfrac{\text{side}}{\sin [\text{opp. angle}]}}$

$$\frac{XZ}{\sin 64° 19'} = \frac{100}{\sin 77° 41'}$$

$$\therefore \quad XZ = \frac{100 \sin 64° 19'}{\sin 77° 41'}$$

$$= 92.243\,428$$

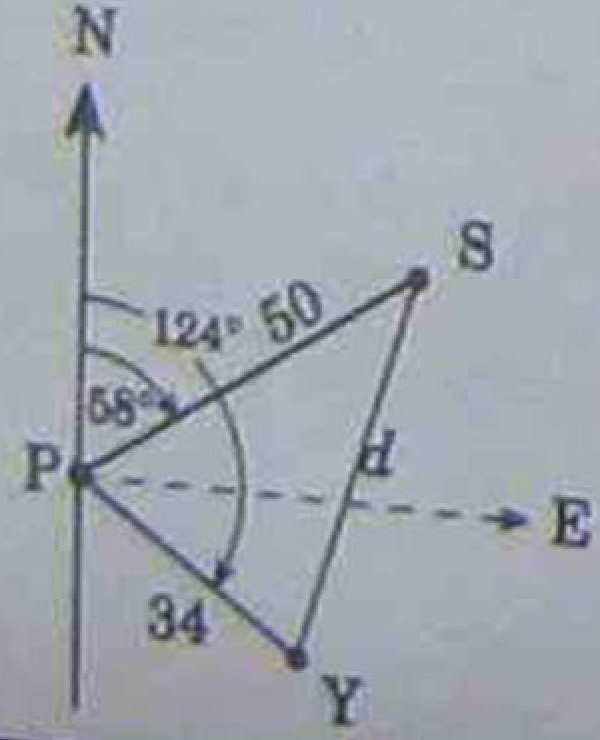
$$\approx 92.2.$$

The distance of X from the target is 92.2 m.

(b) A ship S leaves port P and travels 50 km on a bearing of 58°, while a yacht Y also leaves P and travels 34 km on a bearing of 124°. Find: (i) the distance between the ship and the yacht; and (ii) the bearing of the yacht as measured from the ship.

SOLUTIONS

(i)



Let the distance SY be d km. (Use the Cosine Rule, as we know two sides and the included angle.)

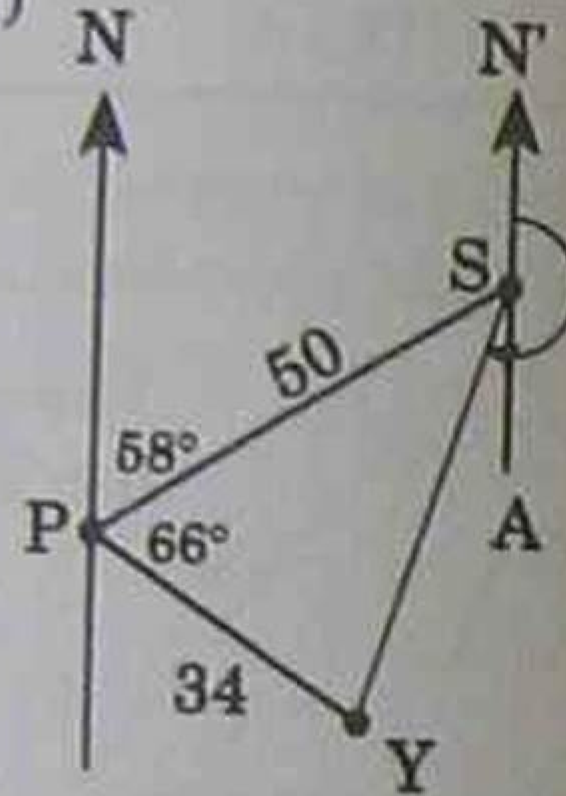
$\angle SPY = 124° - 58° = 66°$.

Then

$$
\begin{aligned}
d^2 &= 50^2 + 34^2 - 2(50)(34)\cos 66° \\
&= 2273.0954 \leftarrow \boxed{\text{Leave on screen}} \\
\therefore\ d &= \sqrt{2273.0954} \\
&= 47.676\,99 \\
&\approx 47.7 \text{ (one decimal place)}
\end{aligned}
$$

The distance between the boats is 47.7 km.

(ii)



The bearing of Y from S is $\angle N'SY$. This can be calculated by subtracting $\angle N'SP$ and $\angle PSY$ from 360°.

Note $\angle N'SY$ is reflex; clockwise direction.

Now $\angle N'SP = 180° - 58° = 122°$ (co-interior angles, $NP \parallel N'A$).

Use the Cosine Rule to calculate $\angle PSY$ as we have three sides of the triangle.

$$\cos \angle PSY = \frac{\left(50^2 + 47.7^2 - 34^2\right)}{\left(2 \times 50 \times 47.7\right)}$$

$$\angle PSY = 40^\circ\, 39'$$

$$\text{Bearing of Y} = 360^\circ - \left(122^\circ + 40^\circ\, 39'\right)$$

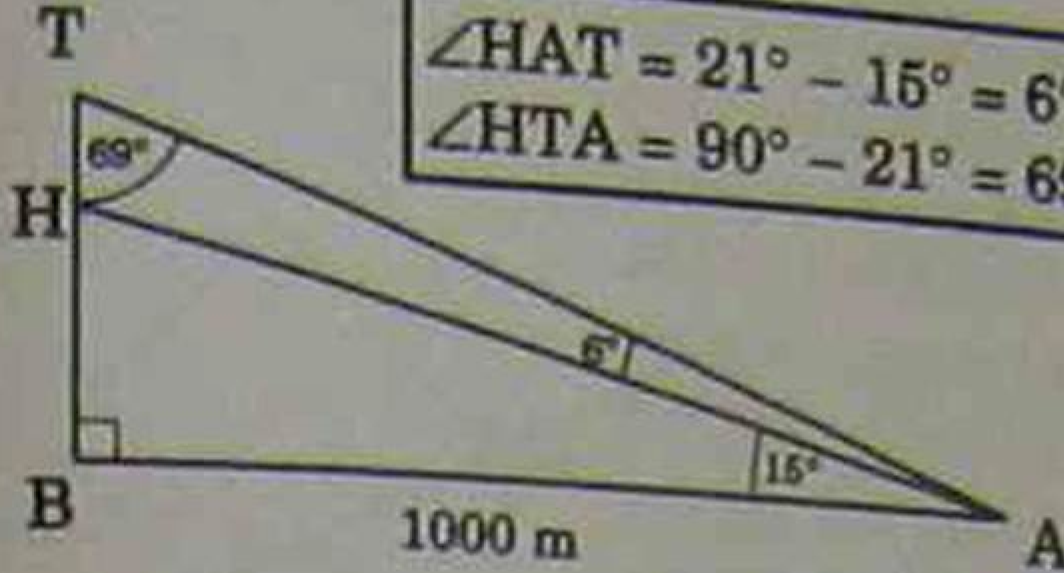
$$= 197^\circ\, 21'$$

The bearing of Y is $197^\circ\, 21'$.

(c) **A two-triangle problem**

From a point, A, level with the base of a hill, the angle of elevation of the top of the hill, H, is measured as 15°. The angle of elevation from A to the top of the tower, T, built on the hill is 21°. If A is 1000 m in a direct line from the base (B) of the hill, calculate the length AH and use this information to calculate the height of the tower (TH) correct to one decimal place.

SOLUTION



$\angle HAT = 21° - 15° = 6°$
$\angle HTA = 90° - 21° = 69°$

In ΔABH,

$$\frac{AB}{AH} = \cos 15°$$

$$\therefore \quad AH = \frac{1000}{\cos 15°}$$
$$= 1035.2762$$

This length is then used as a link to ΔAHT.

In ΔAHT,

$$\frac{TH}{\sin 6^\circ} = \frac{AH}{\sin 69^\circ}$$

$$TH = \frac{AH \sin 6^\circ}{\sin 69^\circ}$$

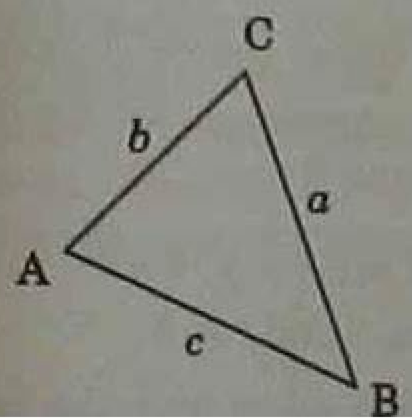
$$= \frac{1035.2762 \times \sin 6^\circ}{\sin 69^\circ}$$

$$= 115.914\,84$$

$$\approx 115.9$$

The height of the tower is 115.9 metres.

10.5.4 The area of a triangle



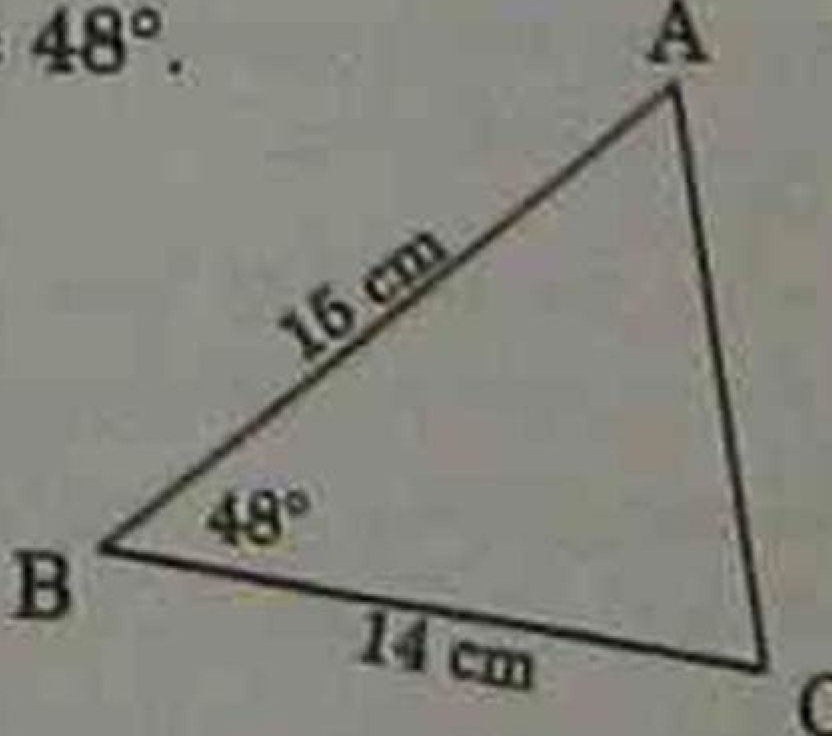
$$\text{Area} = \tfrac{1}{2}ab\sin C$$

Two sides and
included angle

A similar formula can be written for any combination of two sides and the angle between them.

Example

(a) Find the area of the triangle ABC where $a = 14$ cm, $c = 15$ cm and $\angle B = 48°$.



$$\text{Area} = \frac{1}{2}ac\sin B$$
$$= \frac{1}{2} \times 14 \times 15 \times \sin 48°$$
$$= 78.030\,207$$
$$\approx 78 \quad \text{(nearest whole number)}$$

Equivalent formula using angle B

The area is $78\ \text{cm}^2$.