

## 2.1 Generalised arithmetic

### Examples

- (a) The sum of  $x$  and  $y$  is:  $x + y$ .

Sum: add

Product: multiply

Difference: subtract

Quotient: divide

- (b) The average of

$a$ ,  $b$  and  $c$  is:  $\frac{a+b+c}{3}$

Average = mean =  $\frac{\text{sum of scores}}{\text{no. of scores}}$



- (c) The number 4 more than  $c$  is:  $c + 4$ .
- (d) The next three consecutive numbers after  $x$  are  $x + 1$ ,  $x + 2$  and  $x + 3$ .

*Note:* In generalised arithmetic, it can be helpful to substitute numbers for the pronumerals.

- (e) If  $y$  is odd, find the next three consecutive odd numbers.

SOLUTION

$$y + 2, y + 4, y + 6$$

All odd, and even, numbers are separated by two.

- (f) Convert:

- (i) \$ $y$  to cents.

SOLUTION

$$100 \times y = 100y$$

$$\$y = 100y \text{ cents.}$$

Try \$7  
 $.7 \times 100$   
i.e. \$7 = 700¢



(ii)  $p$  litres to mL.

SOLUTION

$$p \times 1000 = 1000p$$

$$p \text{ litres} = 1000p \text{ mL.}$$

Try 8 litres.

$$\therefore 8 \times 1000 \\ = 8000 \text{ mL}$$

(iii)  $y$  minutes to hours.

SOLUTION

$$y \div 60 = \frac{y}{60}$$

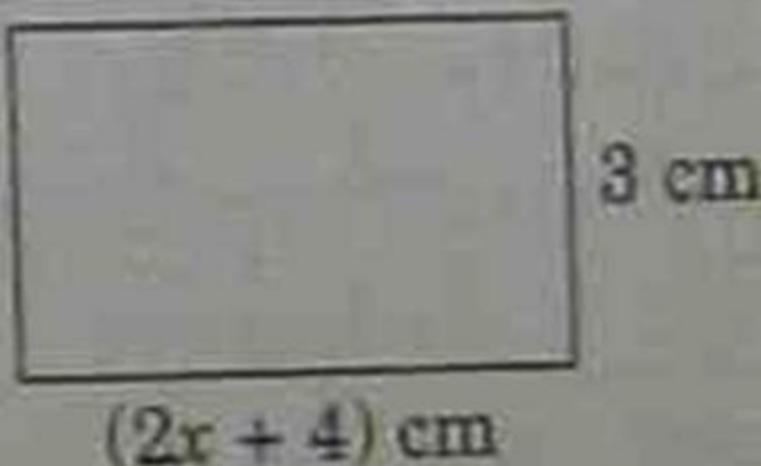
Try 120 minutes.

$$\therefore \frac{120}{60} = 2$$

$$y \text{ minutes} = \frac{y}{60} \text{ hours}$$



(g) Find the area of the rectangle:



### SOLUTION

$$\begin{aligned} A &= (2x + 4) \times 3 \\ &= 3(2x + 4) \end{aligned}$$

Area is  $3(2x + 4)$  cm<sup>2</sup>.



(h) Find the change from \$5 if  $y$  cakes are purchased at  $k$  cents each.

SOLUTION

$$\begin{aligned}\text{Cost} &= y \times k \text{ or } k \times y \\ &= yk \text{ cents or } ky \text{ cents.}\end{aligned}$$

$$\text{Change} = 500 - ky$$

Change is  $(500 - ky)$  cents.



## 2.2 Substitution into algebraic expressions

### Examples

(a) If  $a = 3$ ,  $b = 4$  and  $c = -5$ , evaluate:

$$\begin{aligned}\text{(i)} \quad ab + c &= 3 \times 4 + (-5) \\ &= 12 - 5 \\ &= 7\end{aligned}$$

$$\begin{aligned}\text{(ii)} \quad b - c &= 4 - (-5) \\ &= 4 + 5 \\ &= 9\end{aligned}$$

$$\begin{aligned}\text{(iii)} \quad \frac{bc - 1}{a} &= \frac{4 \times -5 - 1}{3} \\ &= \frac{-20 - 1}{3} \\ &= \frac{-21}{3} \\ &= -7\end{aligned}$$

$$\begin{aligned}\text{(iv)} \quad c(a - b) &= -5(3 - 4) \\ &= -5(-1) \\ &= 5\end{aligned}$$

$$\begin{aligned}\text{(v)} \quad b^2 + c^2 &= (4)^2 + (-5)^2 \\ &= 16 + 25 \\ &= 41\end{aligned}$$

$$\begin{aligned}\text{(vi)} \quad 2c^2 - (2c)^2 &= 2(-5)^2 - (2 \times -5)^2 \\ &= 2(25) - (-10)^2 \\ &= 50 - 100 \\ &= -50\end{aligned}$$



## 2.3 Simplifying algebraic expressions

**Examples:** Expressions are simplified as follows:

$$(a) \quad 3x + 5x + 12x = 20x$$

$$(b) \quad 4x - 2y + 3x + 4y = 7x + 2y$$

$$(c) \quad 12xy - 3yx = 12xy - 3xy \\ = 9xy$$

Remember:  $ab = ba$

$$(d) \quad 4a \times (-2b) = -8ab$$

$$(e) \quad (-6y)^2 = -6y \times -6y \\ = 36y^2$$

$$(f) \quad cd + c = \frac{cd}{c} \\ = \frac{1 \cancel{c} \times d}{\cancel{c}_1} \\ = \frac{d}{1} \\ = d$$

(g)

$$12ab + 3a = \frac{12ab}{3a} \\ = \frac{4 \cancel{12} \times \cancel{a}^1 \times b}{1 \cancel{3} \times \cancel{a}_1} \\ = \frac{4 \times b}{1} \\ = 4b$$

(h)

$$5pq + p^2q = \frac{5pq}{p^2q} \\ = \frac{5 \times \cancel{p}^1 \times \cancel{q}^1}{\cancel{1} \cancel{p} \times p \times \cancel{q}_1} \\ = \frac{5 \times 1}{1 \times p} \\ = \frac{5}{p}$$

(i)

$$\frac{12a + 3a}{5} = \frac{15a}{5} \\ = \frac{15^3 \times a}{\cancel{5}_1} \\ = 3a$$



## 2.4 Simple algebraic fractions

### 2.4.1 Addition and subtraction

Find a common denominator and then add or subtract the numerators.

#### Chapter 2 ◆ ALGEBRA AND QUADRATIC EQUATIONS

**Examples:** Fractions are added or subtracted as follows:

$$(a) \quad \frac{5y}{2} + \frac{y}{3} = \frac{15y}{6} + \frac{2y}{6} \\ = \frac{17y}{6}$$

- Lowest common denominator of 2 and 3 is 6
- 2 times 3 is 6  
 $\therefore 5y \times 3 = 15y$   
 and so on ...

$$(b) \quad \frac{3x}{4} - \frac{x}{2} = \frac{3x}{4} - \frac{2x}{4} \\ = \frac{x}{4}$$

$$(c) \quad \frac{4}{x} - \frac{3}{2x} = \frac{8}{2x} - \frac{3}{2x} \\ = \frac{5}{2x}$$

$$(d) \quad \frac{5}{2y} + \frac{3}{5y} = \frac{25}{10y} + \frac{6}{10y} \\ = \frac{31}{10y}$$



## 2.4.2 Multiplication and division

To multiply: Cancel and then multiply numerators and denominators.

To divide: Find the reciprocal of the second fraction (that is, turn it upside down) and then multiply.

**Examples:** Fractions are multiplied or divided as follows:

(a)  $\frac{x}{3} \times \frac{9}{2x} = \frac{x^1}{3_1} \times \frac{9^3}{2x_1}$   
 $= \frac{3}{2}$   
 $= 1\frac{1}{2}$

(b)  $\frac{4y}{7} \times \frac{21}{6y} = \frac{2^2 4x^1}{7_1} \times \frac{21^1}{1_1 6y_1}$   
 $= \frac{2}{1}$   
 $= 2$

(c)  $\frac{x+2}{6} \times \frac{18}{x+2} = \frac{x+2^1}{6_1} \times \frac{18^3}{x+2_1}$

(d)  $\frac{ab}{4} + \frac{a}{6} = \frac{1^1 ab}{4_2} \times \frac{6^3}{a_1}$   
 $= \frac{3b}{2}$

(e)  $\frac{4x}{5ab} \div \frac{12}{10b} = \frac{1^1 4x}{5ab_1} \times \frac{2^2 10b^1}{12_3}$   
 $= \frac{2x}{3a}$



## 2.5 Removing grouping symbols

The term outside the grouping symbols multiplies the contents of the grouping symbols.

### Examples

(a) Expanding and simplifying:

$$\begin{aligned}\text{(i)} \quad 3(2x + 5y) &= 3 \times 2x + 3 \times 5y \\ &= 6x + 15y\end{aligned}$$

$$\begin{aligned}\text{(ii)} \quad a(2a - 7) &= a \times 2a + a \times -7 \\ &= 2a^2 - 7a\end{aligned}$$

$$\begin{aligned}\text{(iii)} \quad -(3 - 4y) &= -1(3 - 4y) \\ &= -1 \times 3 - 1 \times (-4y) \\ &= -3 + 4y\end{aligned}$$

Note: - sign before the grouping symbols has the effect of negating the contents of grouping symbols.

$$\begin{aligned}\text{(iv)} \quad 5(2x - 4) - 3(5 - x) &= 10x - 20 - 15 + 3x \\ &= 13x - 35\end{aligned}$$

$$\begin{aligned}\text{(v)} \quad x(x + 3) - 2(x + 3) &= x^2 + 3x - 2x - 6 \\ &= x^2 + x - 6\end{aligned}$$



(b)

Simplifying fractions:

$$\begin{aligned}
 \text{(i)} \quad & \frac{x+2}{3} + \frac{x-4}{4} = \frac{4(x+2)}{12} + \frac{3(x-4)}{12} \\
 & \qquad \qquad \qquad \boxed{\text{Good to include this step with grouping symbols}} \\
 & = \frac{4(x+2) + 3(x-4)}{12} \\
 & = \frac{4x+8+3x-12}{12} \\
 & = \frac{7x-4}{12}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad & \frac{x}{5} - \frac{x+1}{2} = \frac{2x}{10} - \frac{5(x+1)}{10} \\
 & = \frac{2x-5(x+1)}{10} \\
 & = \frac{2x-5x-5}{10} \\
 & = \frac{-3x-5}{10}
 \end{aligned}$$

(c)

Find the difference between  $7x^2 - 4x$  and  $3x + 5x^2$ .**SOLUTION**

$$\begin{aligned}
 & 7x^2 - 4x - (3x + 5x^2) \\
 & = 7x^2 - 4x - 3x - 5x^2 \\
 & = 2x^2 - 7x
 \end{aligned}$$



## 2.6 Binomial products

---

A binomial expression has two terms, for example  $2x + 1$ , so a binomial product is the result of multiplying two binomial expressions.

Each term in the first binomial expression multiplies each term in the second expression.

---

Examples:



**Examples:**

Expand and simplify:

(a) 
$$(x + 2)(x + 4)$$

$$= x(x + 4) + 2(x + 4)$$

$$= x^2 + 4x + 2x + 8$$

$$= x^2 + 6x + 8$$

(b) 
$$(x - 2y)(x - 3y)$$

$$= x(x - 3y) - 2y(x - 3y)$$

$$= x^2 - 3xy - 2yx + 6y^2$$

$$= x^2 - 5xy + 6y^2$$

(c) 
$$(a - b)(a + b)$$

$$= a(a + b) - b(a + b)$$

$$= a^2 + ab - ba - b^2$$

$$= a^2 - b^2$$

This method can be shortened of course, or other methods used, such as:

- The Robin Hood method (with arrows ...).

**Example:** Expand and simplify  $(x + 4)(x - 3)$

$$(x + 4)(\cancel{x} - \cancel{3}) = x^2 - 3x + 4x - 12$$

$$= x^2 + x - 12$$

- The FOIL method (First, Outside, Inside, Last)

**Example:** Expand and simplify  $(3x + 1)(2x + 5)$

$$(3x + 1)(2x + 5)$$

$$= (3x)(2x) + (3x)(5) + (1)(2x) + (1)(5)$$

↑              ↑              ↑              ↑  
first        outside     inside        last

$$= 6x^2 + 15x + 2x + 5$$

$$= 6x^2 + 17x + 5.$$



## 2.6.1 Special products

The following results are very important and must be known for success in Year 10:

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a - b)(a + b) = a^2 - b^2$$

**Examples:** Expanding and simplifying:

$$\begin{aligned}(a) \quad (x + 5)^2 &= (x)^2 + 2(x)(5) + (5)^2 \\&= x^2 + 10x + 25\end{aligned}$$

$$\begin{aligned}(b) \quad (a - 2)^2 &= (a)^2 + 2(a)(-2) + (-2)^2 \\&= a^2 - 4a + 4\end{aligned}$$

$$\begin{aligned}(c) \quad (3x - 4)^2 &= (3x)^2 + 2(3x)(-4) + (-4)^2 \\&= 9x^2 - 24x + 16\end{aligned}$$

$$\begin{aligned}(d) \quad (d - 2)(d + 2) &= (d)^2 - (2)^2 \\&= d^2 - 4\end{aligned}$$

$$\begin{aligned}(e) \quad (5y + 3)(5y - 3) &= (5y)^2 - (3)^2 \\&= 25y^2 - 9\end{aligned}$$

$$\begin{aligned}(f) \quad (x^2 - y^2)(x^2 + y^2) &= (x^2)^2 - (y^2)^2 \\&= x^4 - y^4\end{aligned}$$



$$= d^2 - 4$$

## 2.6.2 Using these special products

**Examples:** Using the special products to evaluate

(a)  $101^2 = (100 + 1)^2$

$$\begin{aligned} &= 100^2 + 2(100)(1) + 1^2 && [(a + b)^2 = a^2 + 2ab + b^2] \\ &= 10\,000 + 200 + 1 \\ &= 10201 \end{aligned}$$

(b)  $98^2 = (100 - 2)^2$

$$\begin{aligned} &= 100^2 - 2(100)(2) + 2^2 && [(a - b)^2 = a^2 - 2ab + b^2] \\ &= 10\,000 - 400 + 4 \\ &= 9604 \end{aligned}$$

### 2.6.3 More-difficult expansions

**Examples:** Expanding and simplifying:

$$\begin{aligned}(a) \quad & (x-3)(x+3) - (x-4)^2 \\&= x^2 - 9 - (x^2 - 8x + 16) \\&= x^2 - 9 - x^2 + 8x - 16 \\&= 8x - 25\end{aligned}$$

$$\begin{aligned}(b) \quad & 2(x+6)^2 - (4-x)(x+4) \\&= 2(x^2 + 12x + 36) - (4-x)(4+x) \\&= 2x^2 + 24x + 72 - (16 - x^2) \\&= 2x^2 + 24x + 72 - 16 + x^2 \\&= 3x^2 + 24x + 56\end{aligned}$$



## 2.7 Factorisations

### 2.7.1 Common factors

We look for the highest, or largest, factor common to the terms in the expression — this is the opposite to expanding.

#### Examples: Factorising

(a)

$$4x - 6 = 2(2x - 3)$$

Check by expanding.

(b)

$$xy - 3x = x(y - 3)$$

(c)

$$12ab - 14a = 2a(6b - 7)$$

(d)

$$3x^2 - 6x - 3 = 3(x^2 - 2x - 1)$$

(e)

$$-12x - 4 = -4(3x + 1)$$

(f)

$$9a^2 + 12a^2b = 3a^2(3 + 4b)$$

(g)

$$\begin{aligned}4x(x + y) + 3(x + y) \\&= (x + y)[4x + 3] \\&= (x + y)(4x + 3)\end{aligned}$$

You must take both 2 and  $a$  as common factors.



## 2.7.2 Factorising by grouping in pairs

We can factorise four-term expressions by grouping in pairs.

Examples: Factorising:

(a)  $ax + bx + ay + by$

$$= x(a + b) + y(a + b)$$

$$= (a + b)[x + y]$$

$$= (a + b)(x + y)$$

(c)  $xy + y^2 - x - y$

$$= y(x + y) - 1(x + y)$$

$$= (x + y)(y - 1)$$

(b)  $p^2 + mq + pq + mp$

$$= p^2 + pq + mq + mp$$

$$= p(p + q) + m(p + q)$$

$$= (p + q)(p + m)$$



### 2.7.3 Difference of two squares

We can reverse an earlier rule:  $a^2 - b^2 = (a - b)(a + b)$

Examples: Factorising:

$$\begin{aligned}(a) \quad c^2 - 9 &= (c)^2 - (3)^2 \\&= (c - 3)(c + 3)\end{aligned}$$

Note: It does not matter if your answer has this order:  
 $(c + 3)(c - 3)$ .

The order of factors is not important,  
that is,  $2 \times 3 = 3 \times 2$ .

$$\begin{aligned}(b) \quad 49 - 4a^2 &= (7)^2 - (2a)^2 \\&= (7 - 2a)(7 + 2a)\end{aligned}$$

$$\begin{aligned}(c) \quad x^4 - 1 &= (x^2)^2 - (1)^2 \\&= (x^2 - 1)(x^2 + 1) \\&= (x - 1)(x + 1)(x^2 + 1)\end{aligned}$$



## 2.7.4 Completing the square

Numbers such as 1, 4, 9, 16, ... are known as square numbers while expressions such as  $(x - 4)^2$ ,  $(x + 5)^2$ , etc. are known as perfect squares.

**Examples:** What must be added to the following expressions to give perfect squares?

(a)  $x^2 - 4x$

↑ We halve the coefficient of  $x$  and square it

that is,  $x^2 - 4x + 4$

as  $\left(\frac{-4}{2}\right)^2 = 4$

∴ we add 4

that is,  $x^2 - 4x + 4 = (x - 2)^2$

$a^2 - 2ab + b^2 = (a - b)^2$

(b)  $x^2 + 6x$

∴  $x^2 + 6x + 9 = (x + 3)^2$

$a^2 + 2ab + b^2 = (a + b)^2$

that is, we add 9

(c)  $x^2 - 5x$

∴  $x^2 - 5x + \left(\frac{-5}{2}\right)^2$

that is,  $x^2 - 5x + \frac{25}{4}$

that is,  $x^2 - 5x + 6\frac{1}{4} = \left(x - \frac{5}{2}\right)^2$

∴ we add  $6\frac{1}{4}$



## 2.7.5 The monic quadratic trinomial

- An expression with three terms is called a *trinomial*.
- A trinomial with a highest power of 2 is called a *quadratic*.
- If the coefficient (number in front of) the term with the power of 2 is 1, the quadratic trinomial is *monic*,  $\therefore x^2 + 5x + 6$  is a monic quadratic trinomial.

As

$$\begin{aligned}(x+a)(x+b) &= x^2 + bx + ax + ab \\ &= x^2 + (a+b)x + ab\end{aligned}$$

then, factorising,  $x^2 + (a+b)x + ab = (x+a)(x+b)$ .

Hence, to factorise  $x^2 + 5x + 6$  we are looking for two numbers that add together to give 5, (that is,  $a + b = 5$ ) and multiply together to give 6 (that is,  $ab = 6$ ).



### Examples: Factorising:

(a)  $x^2 + 5x + 4$   
(that is,  $a + b = 5$ ,  $ab = 4$ )  
 $= (x + 4)(x + 1)$

(b)  $x^2 - 5x + 6$   
(that is,  $a + b = -5$ ,  $ab = 6$ )  
 $= (x - 3)(x - 2)$

(c)  $x^2 - 3x - 4$   
(that is,  $a + b = -3$ ,  $ab = -4$ )  
 $= (x - 4)(x + 1)$

(d)  $x^2 + 5x - 14$   
(that is,  $a + b = 5$ ,  $ab = -14$ )  
 $= (x + 7)(x - 2)$



Another helpful rule is:

- For  $x^2 + 5x + 4$   
 $\uparrow\uparrow$  positive here, we say 'both' 'this sign'; that is, **both positive**.
- For  $x^2 - 3x - 4$   
 $\uparrow \quad \uparrow$  negative here, we say 'the bigger one is' 'this sign'; that is, **bigger one is negative**.

Examples: Factorising:

(e)  $x^2 + 7x + 12 = (x + 4)(x + 3)$



both positive

(f)  $x^2 - 8x + 15 = (x - 5)(x - 3)$



both negative

(g)  $x^2 - 4x - 12 = (x - 6)(x + 2)$

The 'bigger one' is negative.

(h)  $x^2 + x - 20 = (x + 5)(x - 4)$

The 'bigger one' is positive.



## 2.7.6 Non-monic quadratic trinomial

Most of the hints given above no longer apply, although:

- the two numbers  $a$  and  $b$  still multiply together ( $ab$ ) to give our constant term;
- if this constant term is positive, we can still say 'both positive/negative'. (See above.)

In  $ax^2 + bx + c$ , the  $c$  is the constant term and is said to be independent of  $x$ .



### Examples:

(a) Factorise  $2x^2 + 13x + 21$ .

SOLUTION

$$2x^2 + 13x + 21 = (2x \quad ) (x \quad ).$$

We can see that they are both positive,

$$\therefore 2x^2 + 13x + 21 = (2x + \quad ) (x + \quad ).$$

Now, it's simply trial and error, and checking by expanding the answer to get back to the expression.

$$\therefore 2x^2 + 13x + 21 = (2x + 7)(x + 3).$$

(b) Factorise  $3x^2 - 2x - 8$ .

SOLUTION

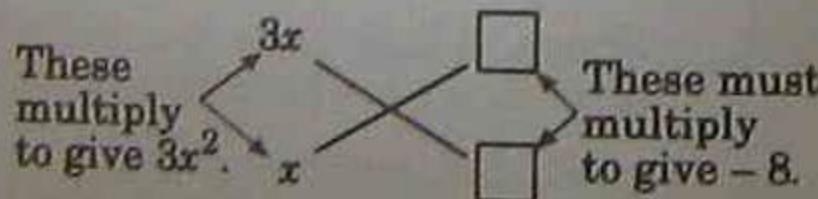
Start with  $(3x \quad )(x \quad )$  and trial and error, knowing that we are looking for two numbers that multiply together to give  $-8$ .

$$\therefore 3x^2 - 2x - 8 = (3x + 4)(x - 2).$$

Now try a second method: This is called the cross method, where the guess and verify stage occurs around a cross.

(c)

Factorise  $3x^2 + 10x - 8$ .



We then multiply across the diagonals and add terms.

Thus:

$$3x^2 \swarrow \begin{array}{c} 3x \\ \times \\ x \end{array} \searrow -4 \qquad \qquad \qquad -8 \swarrow \begin{array}{c} -4 \\ \times \\ 2 \end{array} \searrow -16$$

$$\begin{aligned}\therefore 3x \times 2 + x \times -4 \\ = 6x - 4x \\ = 2x.\end{aligned}$$

No! It must be  $10x$ .

Try again:



$$3x^2 \swarrow \begin{array}{c} 3x \\ \times \\ x \end{array} \begin{array}{c} -2 \\ \times \\ 4 \end{array} \searrow -8$$

$$\begin{aligned}\therefore 3x \times 4 + x \times -2 \\ = 12x - 2x \\ = 10x.\end{aligned}$$

Yes,

$$\therefore (3x - 2)(x + 4).$$

(d) Factorise  $4x^2 - 16x + 15$ .

**SOLUTION:** Note that the  $4x^2$  means we could choose  $4x$  and  $x$  or  $2x$  and  $2x$ .

Once again, by trial and error:

$$4x^2 \swarrow \begin{array}{c} 2x \\ \times \\ 2x \end{array} \begin{array}{c} -5 \\ \times \\ -3 \end{array} \searrow 15$$

$$\begin{aligned}2x \times -3 + 2x \times -5 \\ = -6x - 10x \\ = -16x,\end{aligned}$$

$$\therefore 4x^2 - 16x + 15 = (2x - 5)(2x - 3).$$

Another method of factorising non-monic quadratic trinomial is as follows:

If the trinomial is of the form  $ax^2 + bx + c$ , we rewrite  $bx$  as two separate terms, which allows us to factorise pairs of terms.



## Examples

(e) Factorise  $2x^2 + 5x - 3$ .

- We multiply  $2x^2$  by  $-3$ , that is,  $-6x^2$ .
- We now look for two factors which add together to give  $5$  (from the  $5x$ ) and multiply together to give  $-6$  (from the  $-6x^2$ ).

Thus the factors are  $6$  and  $-1$ .

That is,

$$\begin{aligned}2x^2 + 5x - 3 &= 2x^2 + 6x - x - 3 \\&= 2x(x + 3) - 1(x + 3) \\&= (x + 3)(2x - 1) \\&= (2x - 1)(x + 3).\end{aligned}$$

(f) Factorise  $8a^2 + 10a - 3$

SOLUTION

$$8a^2 \times -3 = -24a^2,$$

therefore look for factors which add to give  $10$  and multiply to give  $-24$ .

That is,  $12$  and  $-2$ .

Therefore

$$\begin{aligned}8a^2 + 10a - 3 &= 8a^2 - 2a + 12a - 3 \\&= 2a(4a - 1) + 3(4a - 1) \\&= (4a - 1)(2a + 3).\end{aligned}$$



A further method is as follows:

### Examples

(g) Factorise  $4a^2 - 13a + 9$ .

#### SOLUTION

- The coefficient of  $a^2$  and the constant term are multiplied ... here  $4 \times 9 = 36$ .
- The coefficient of  $a^2$  (here it is 4) is put in both brackets and as a denominator.

Thus,

$$4a^2 - 13a + 9 = \frac{(4a \quad ) (4a \quad )}{4}$$

36

- We now look for two numbers that add together to get  $-13$  (from  $-13a$ ), and multiply together to get  $36$ .

That is,  $-9$  and  $-4$ .

Therefore:

$$4a^2 - 13a + 9 = \frac{(4a - 9)(4a - 4)}{4}$$

- We then cancel ... here  $4a - 4$  is divided by 4 to get  $a - 1$ .

Thus  $4a^2 - 13a + 9 = (4a - 9)(a - 1)$ .

