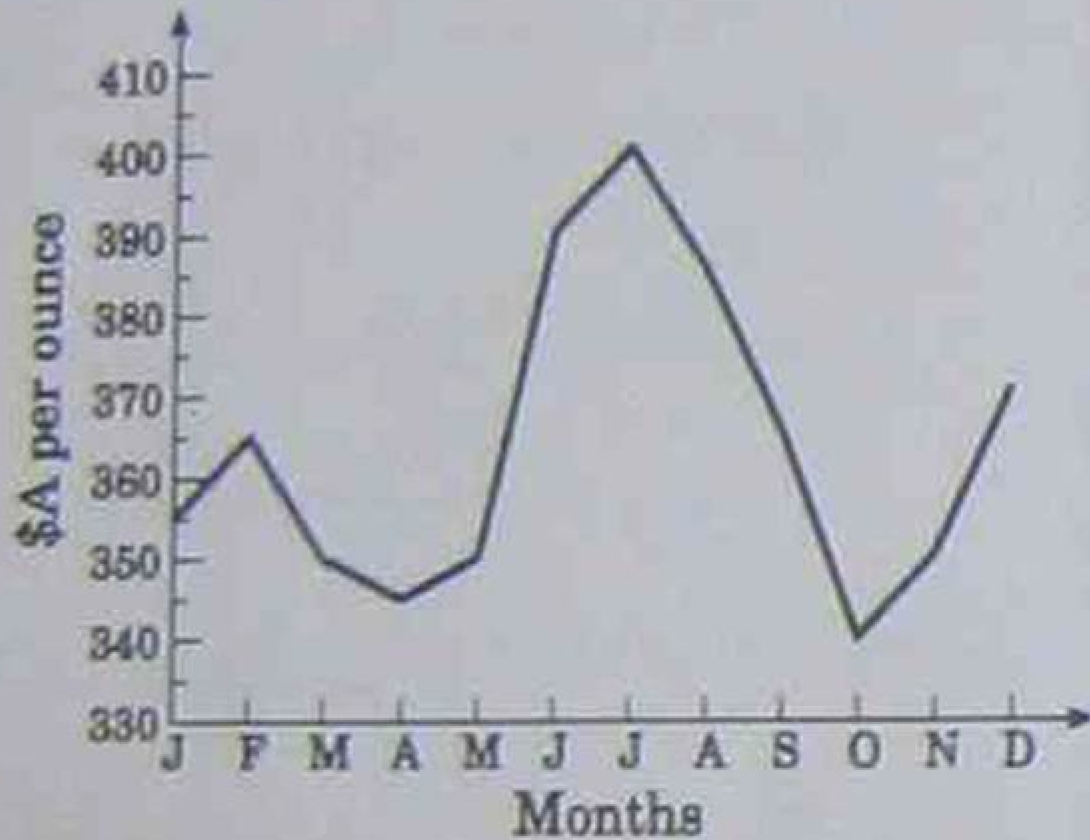
Examples

(a)



The above graph shows the variations in the price of gold as recorded at the beginning of each month from 1 January to 1 December. Use the graph to answer the following questions:

(i) What does the smallest marked unit on the vertical axis represent?

ANSWER: $5 (Australian) per ounce.

(ii) What was the price of gold on 1 March?

ANSWER: $350

(iii) When was the price of gold at its highest point, and what was the price?

ANSWER: 1 July; $400.

(iv) During which month did the greatest price rise occur and how much did it rise?

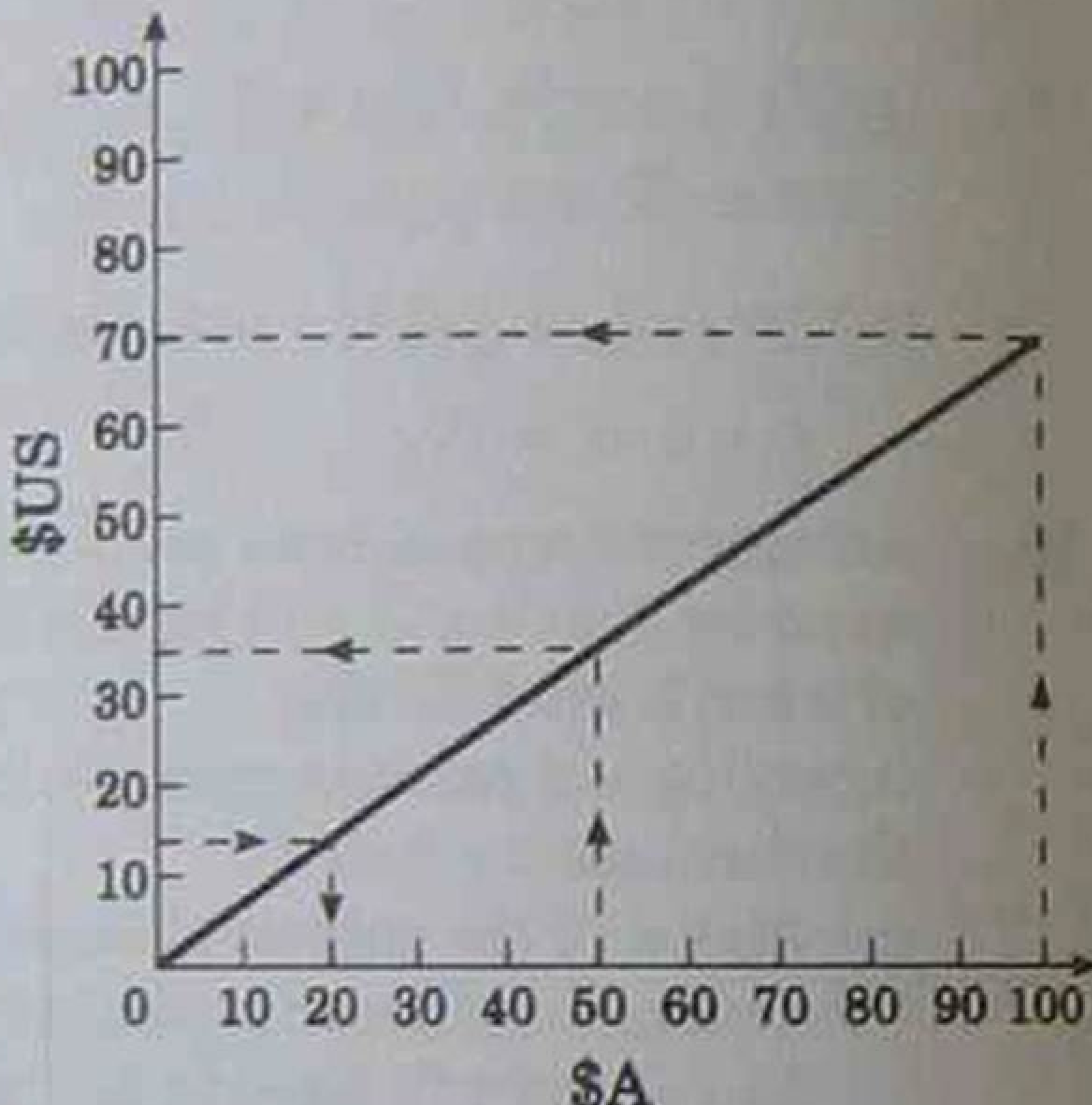
ANSWER: May; $40

Steepest part of graph

(v) When was the price of gold $350 per ounce?

ANSWER: March, May, November.

(b)



The above graph is used to convert Australian dollars to US dollars. From the conversion graph we can see that \$100 (Australian) converts to \$70 (United States), that is, \$A100 = \$US70. Use the graph to convert:

(i) \$A50 to \$US

ANSWER: \$A50 = \$US35.

(ii) \$US14 to \$A

ANSWER: \$US14 = \$A20

(c) Find a formula for \$US (S) in terms of \$A (A), by using the information on the graph.

SOLUTION: Take points from the graph: (0, 0) and (100, 70)

Use coordinate geometry formulae, that is,

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

Thus, $$\frac{S-0}{A-0}=\frac{70-0}{100-0},$$

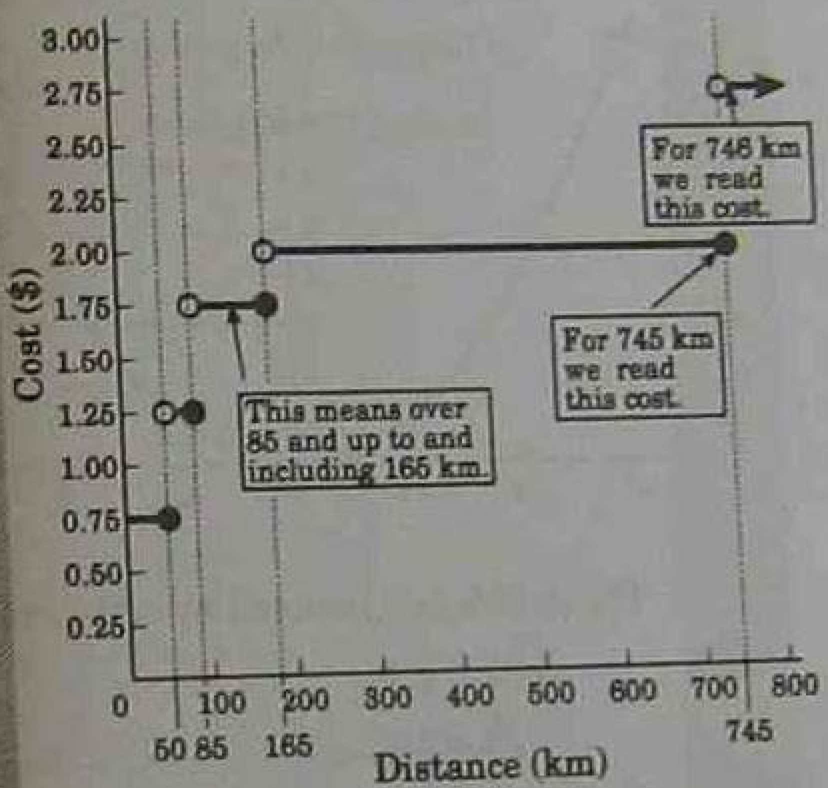
therefore, $$\frac{S}{A}=\frac{7\cancel{0}}{10\cancel{0}}=\frac{7}{10}$$

and $$S=\frac{7A}{10}.$$

5.2 Other types of line graphs

5.2.1 Step graphs

Example: The graph below indicates telephone charges for STD calls lasting five minutes (or part thereof).



Use the graph to calculate the cost of the following calls:

(i) Five minutes over a distance of 100 km.

ANSWER: $1.75.

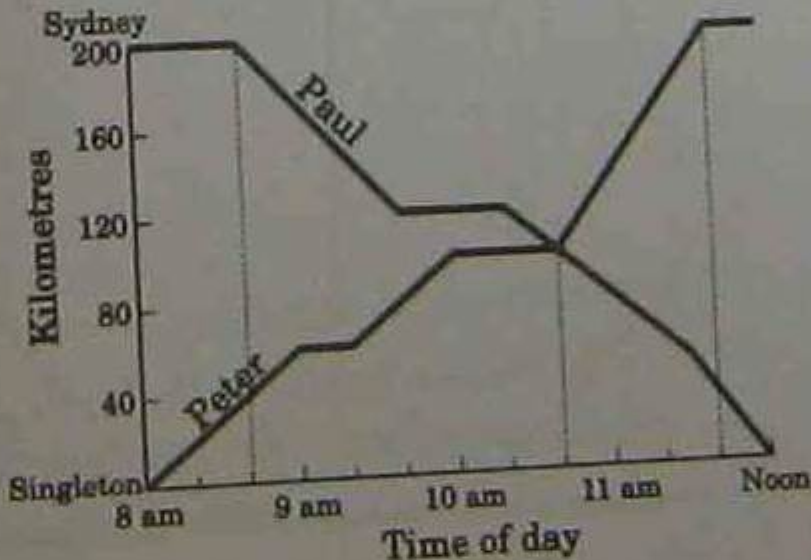
(ii) Ten minutes over a distance of 85 km.

ANSWER: $2 \times \$1.25 = \$2.50$.

10 minutes is 2×5 minutes

5.2.2 Travel graphs

Example



The above graph shows the travel patterns of Peter who travelled from Singleton to Sydney and Paul who travelled from Sydney to Singleton.

(i) How long did Peter travel for before he stopped?

ANSWER: 1 hour

(ii) What was his average speed before he stopped?

ANSWER: 60 km/h

(iii) At what time did Peter arrive in Sydney?

ANSWER: 11:40 a.m.

(iv) At what time did Paul leave Sydney?

ANSWER: 8:40 a.m.

(v) At what time did Peter meet Paul on the road?

ANSWER: 10:40 a.m.

Continued

(vi) What was Paul's average speed for the entire trip?

SOLUTION: From 8:40 a.m. to 12 noon
$=3$ h 20 min $= 3\frac{1}{3}$ hours.

$$\text{Speed} = \frac{\text{distance}}{\text{time}}$$
$$= \frac{200}{3\frac{1}{3}}$$
$$= 60 \text{ km/h}$$

5.3 The parabola

5.3.1 The graph of $y = x^2$

To graph any equation which results in a curve we must plot points to make a clear shape.

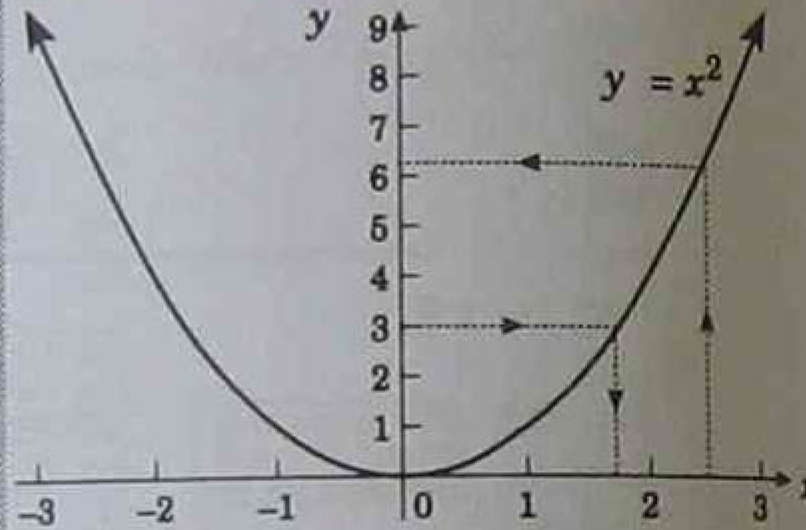
Example: For $y = x^2$

x	–3	–2	–1	0	1	2	3
y	9	4	1	0	1	4	9

Our points are (–3, 9), (–2, 4), (–1, 1), (0, 0), (1, 1), (2, 4), (3, 9)

The y-axis is the axis of symmetry. The curve is symmetrical about the y-axis.

The curve is always above the x--axis because both positives and negatives squared give positive answers.

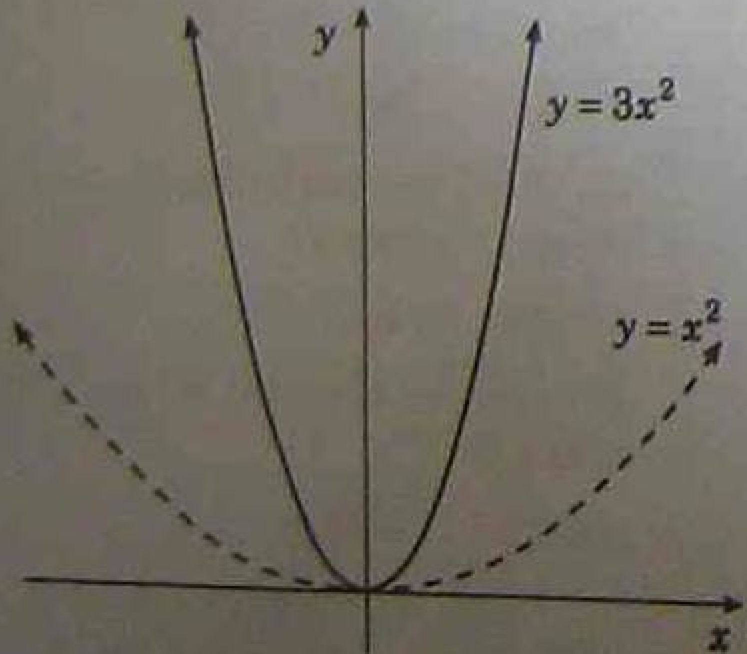


The graph can be used to estimate:

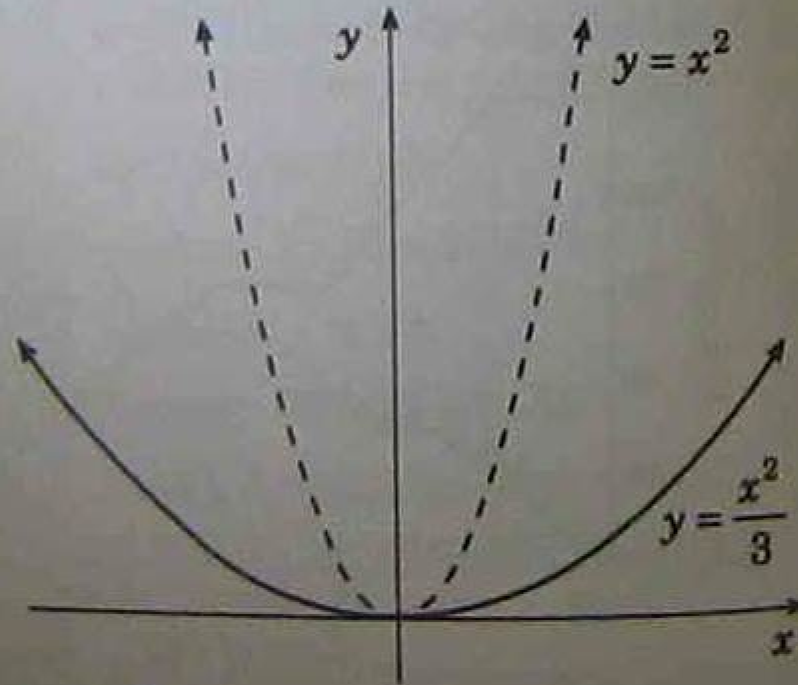
$2.5^2 \approx 6.2$,

$\sqrt{3} \approx 1.7$.

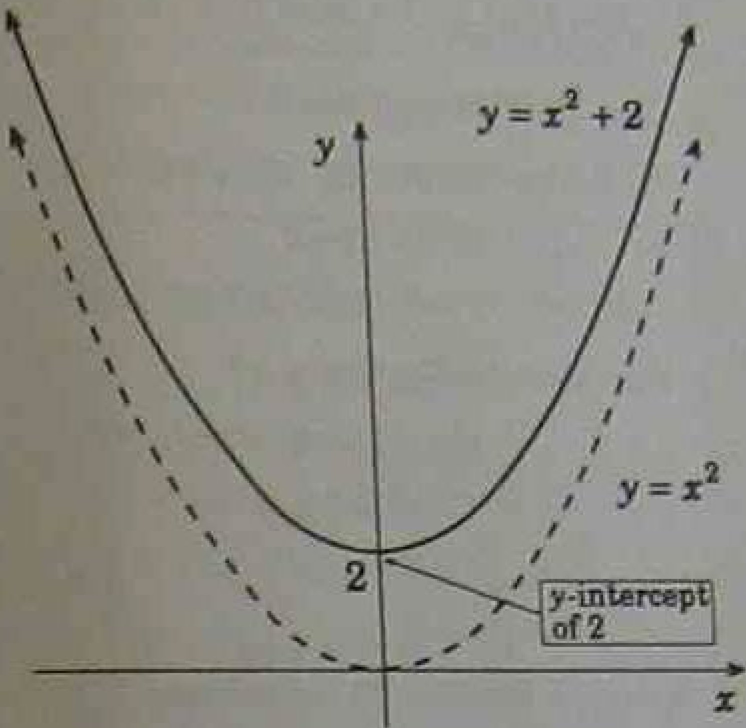
5.3.2 Graphing $y = ax^2 + c$, $y = (x-k)^2$, etc



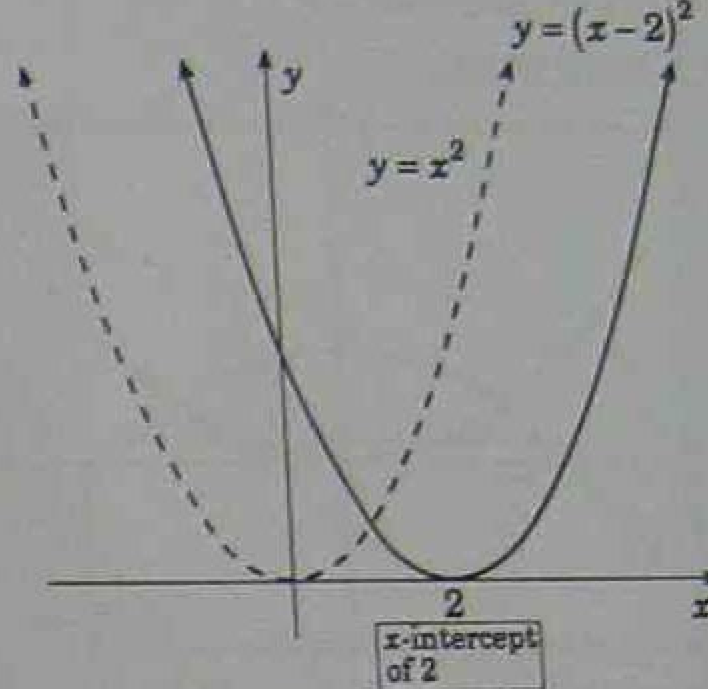
$y = 3x^2$ is 'inside' $y = x^2$



$y = \frac{x^2}{3}$ (or $y = \frac{1}{3}x^2$) is 'outside' $y = x^2$



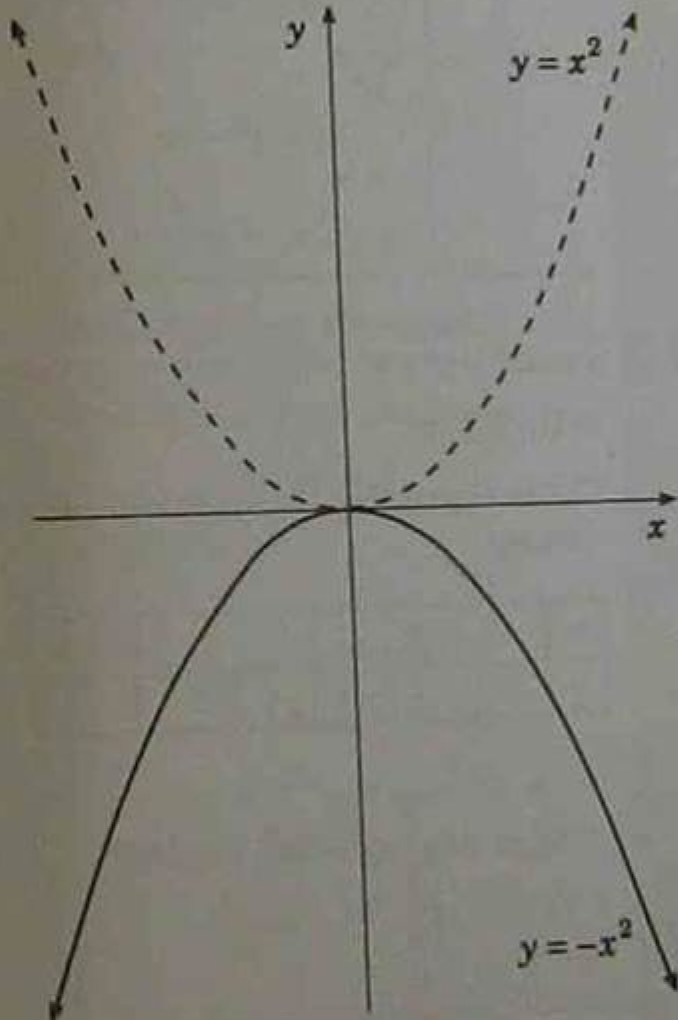
$y = x^2 + 2$ is 'shifted up' two units from $y = x^2$



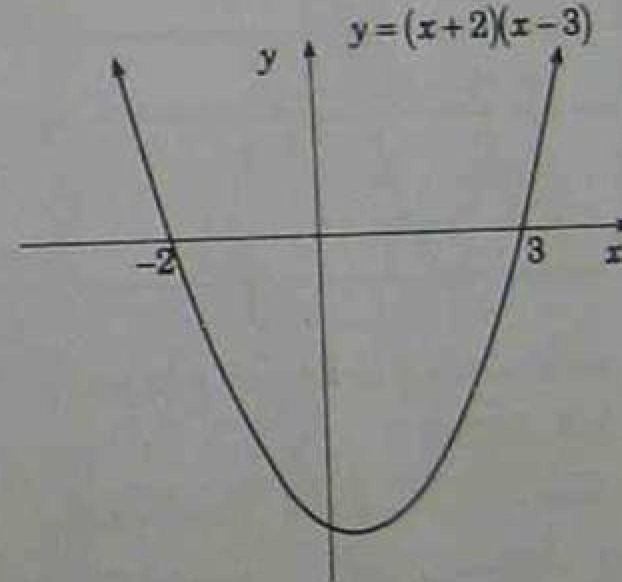
$y = (x-2)^2$ is shifted to right two units from $y = x^2$

two units from $y=x^2$

$y=(x-2)^2$ is shifted to right two units from $y=x^2$



$y=-x^2$ is 'upside down' view of $y=x^2$, that is coefficient of x^2 is negative, therefore concave down, not concave up.

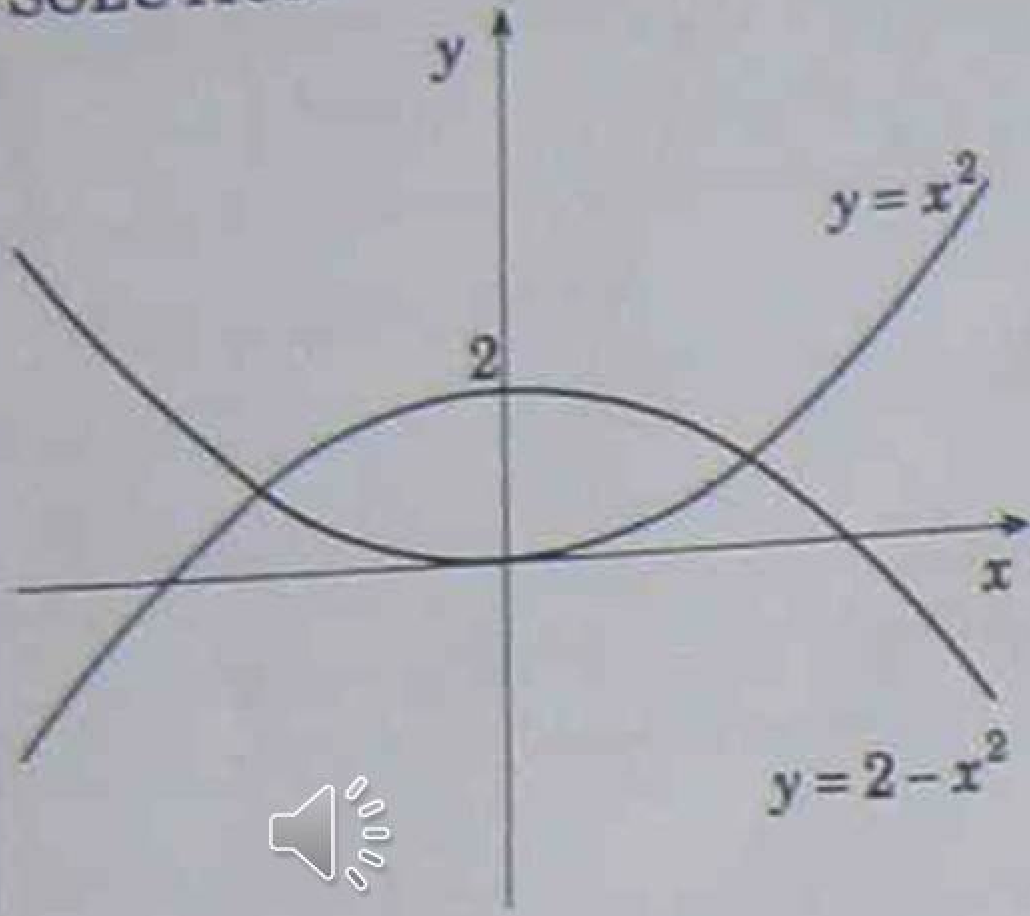


When $x=0$, $x=-2$ and $x=3$ give us the x-intercepts.

Examples: Graph the following parabolas on same number plane:

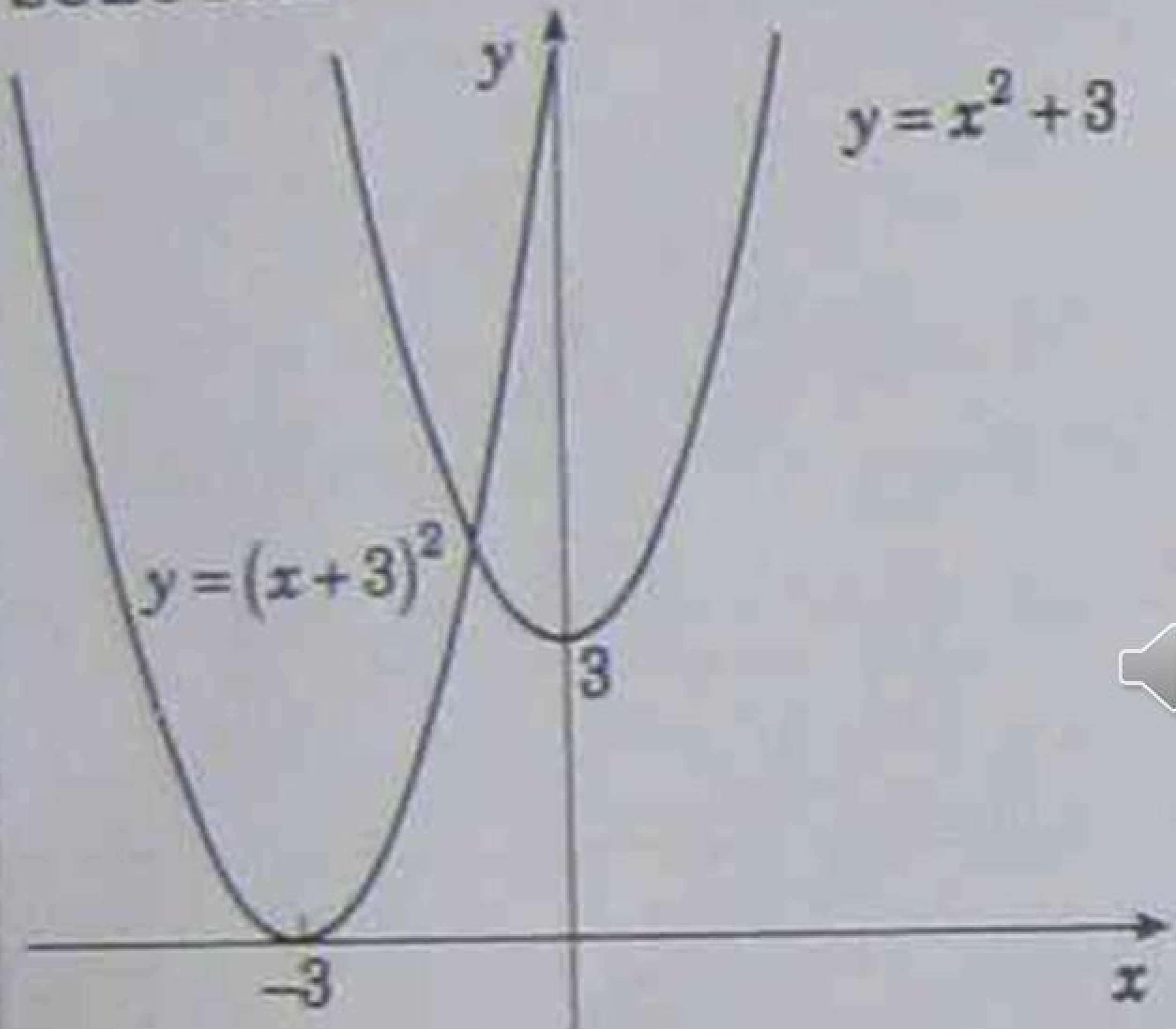
(a) $y = x^2$ and $y = 2 - x^2$

SOLUTION



(b) $y = x^2 + 3$ and $y = (x + 3)^2$

SOLUTION

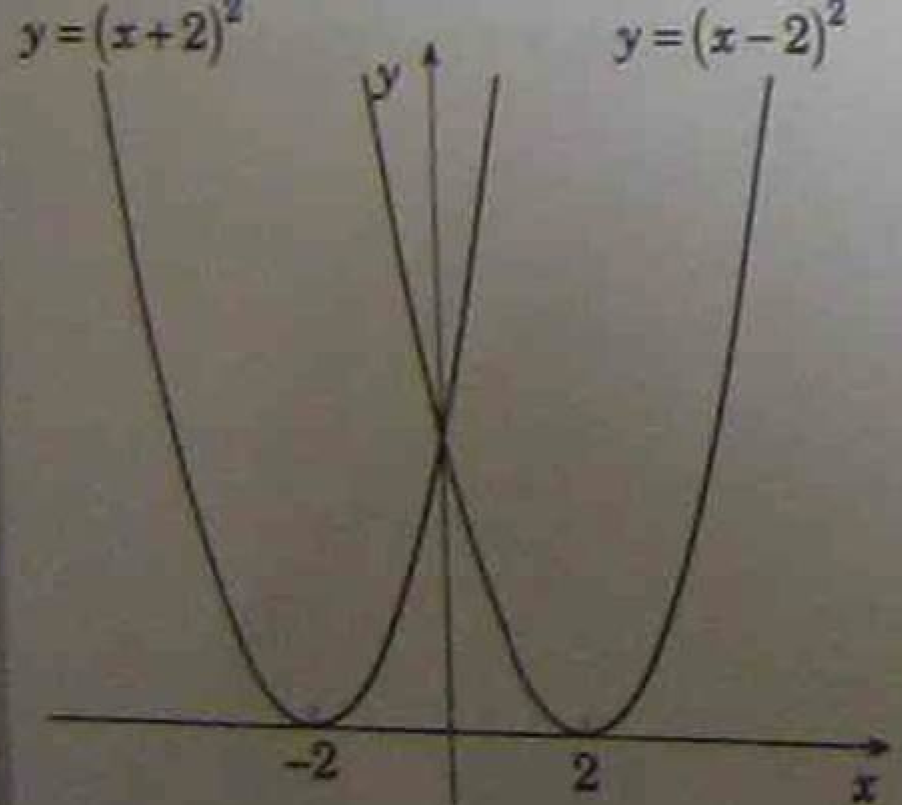


(c) $y = x^2 - 4x + 4$ and $y = x^2 + 4x + 4$

SOLUTION

Now, $y = x^2 - 4x + 4 \quad y = x^2 + 4x + 4$

$\therefore \quad y = (x-2)^2 \quad y = (x+2)^2$



(d) $y = (x + 2)(x - 3)$ and $y = x^2 - 7x$

SOLUTION

For $y = (x+2)(x-3)$

to find x - intercept, let $y = 0$

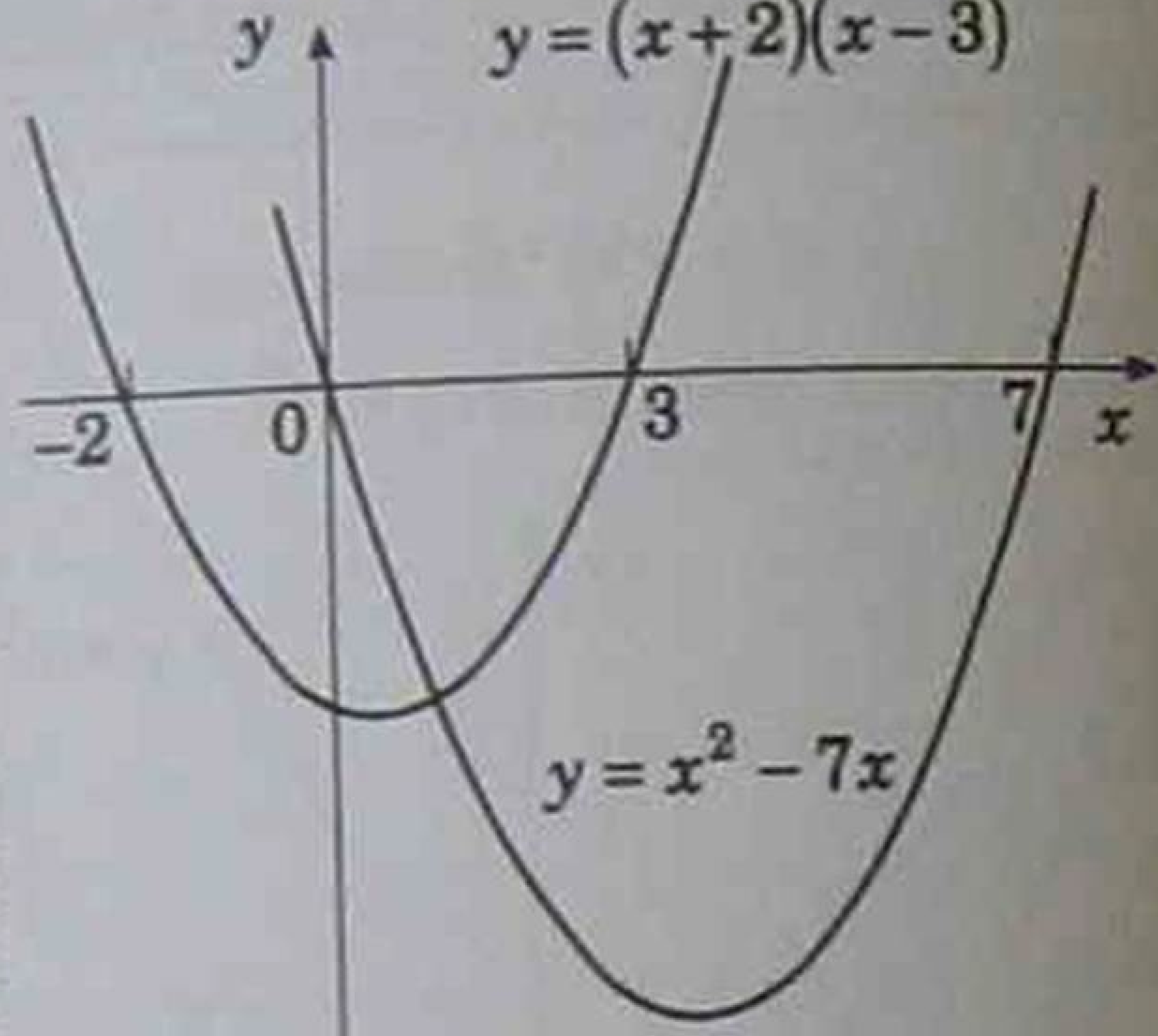
$$\therefore \quad (x+2)(x-3) = 0$$

$$x = -2, 3$$

For $y = x^2 - 7x$, let $y = 0$

$$\therefore \quad x(x-7) = 0$$

$$x = 0, 7$$

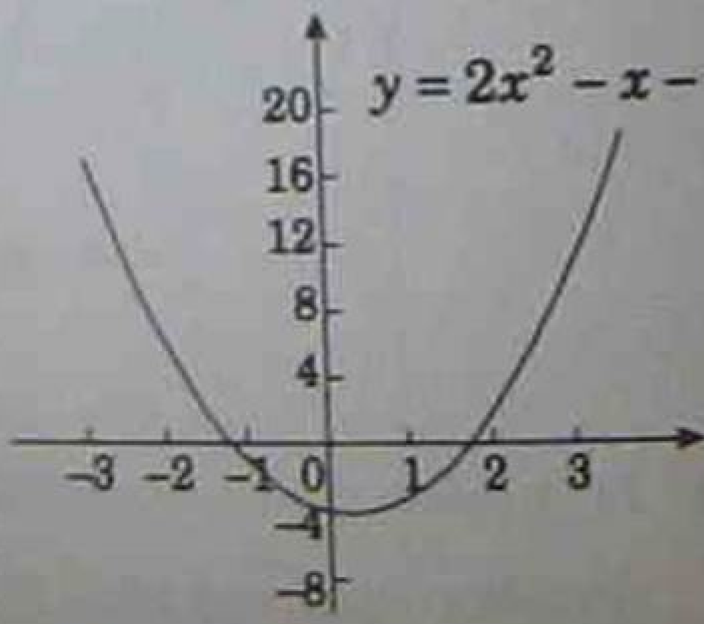

y
$y=(x+2)(x-3)$
-2
0
3
7
x
$y=x^2-7x$

(e) $y = 2x^2 - x - 4$

SOLUTION

Cannot factorise, so must use table of values:

x	-3	-2	-1	0	1	2	3
y	17	6	-1	-4	-3	2	11



5.3.3 The vertex of a parabola

Three methods can be used:

(a) Midpoint of x-intercepts

Example: Find the vertex of

$$y = (x + 3)(x - 1)$$

SOLUTION

x - intercept when $y = 0$

$\therefore \quad (x+3)(x-1) = 0$

$$x = -3, 1$$

$\therefore$ midpoint of -3 and 1

$$= \frac{-3+1}{2}$$

$$= -1$$

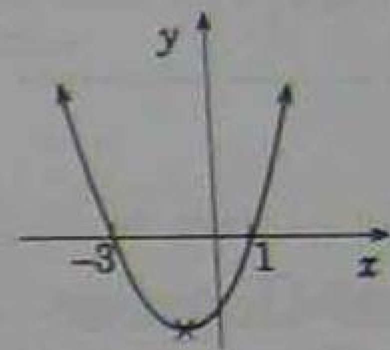
Now substitute in y,

that is, $\quad y = (-1+3)(-1-1)$

$$= (2)(-2)$$

$$= -4$$

$\therefore$ vertex is $(-1, -4)$



The parabola has minimum vaue (as $a > 0$) of -4.

(b) Completing the square

Example: Find the vertex of

$$y = x^2 - 4x + 3$$

SOLUTION

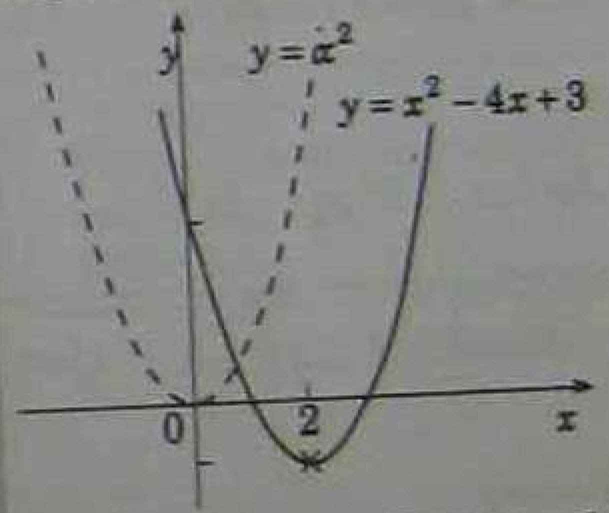
$$y = x^2 - 4x + 3$$

$$\therefore \quad y = x^2 - 4x \quad + 3$$

$$\therefore \quad y = x^2 - 4x + 4 + 3 - 4$$

(We have added a 4, so subtract 4.)

$$\therefore \quad y = (x-2)^2 - 1$$



$y = x^2$ moved across 2 units and down 1 unit. Vertex is (2, –1). The parabola has minimum value of –1.

(c) Using $x = \frac{-b}{2a}$ (axis of symmetry)

Example: Find the vertex of

$$y = -x^2 - 3x + 7$$

SOLUTION

As $y = -x^2 - 3x + 7$ is in form of

$y = ax^2 + bx + c \therefore \quad a = -1, b = -3$

$\therefore \quad x = \frac{-b}{2a}$

$= \frac{3}{-2}$

$= -1\frac{1}{2}$

[$\therefore$ axis of symmetry is $x = -1\frac{1}{2}$]

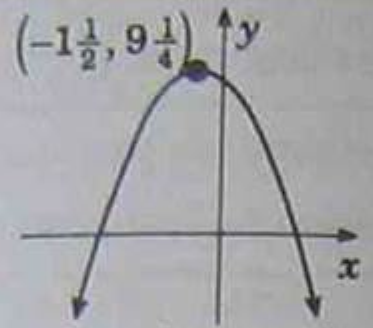
Substitute $x = -1\frac{1}{2}$ in

$$y = -x^2 - 3x + 7$$

$$\therefore \quad y = -\left(1\tfrac{1}{2}\right)^2 - 3\left(-1\tfrac{1}{2}\right) + 7$$

$$= -2\tfrac{1}{4} + 4\tfrac{1}{2} + 7$$

$$= 9\tfrac{1}{4}$$



$\therefore$ vertex is $\left(-1\frac{1}{2}, 9\frac{1}{4}\right)$

[the parabola has a maximum value (as $a < 0$) of $9\frac{1}{4}$]

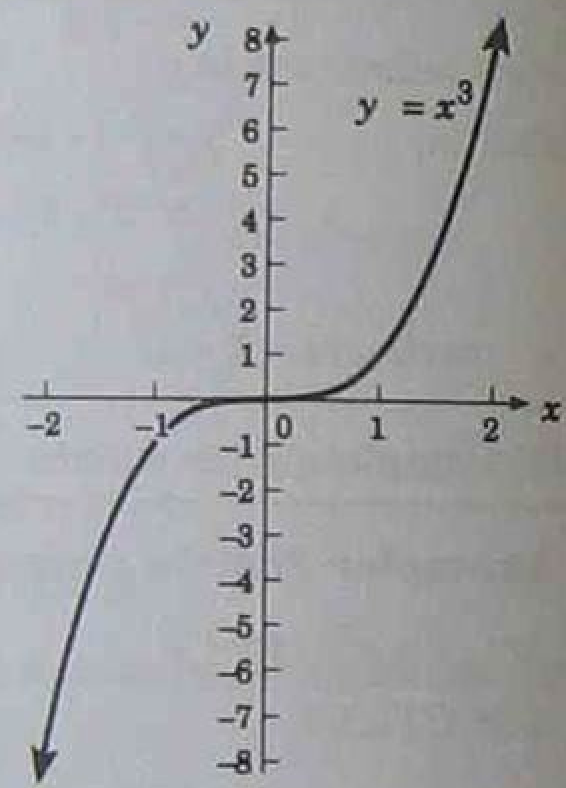
5.4 Other Graphs

5.4.1 The cubic (for example $y = x^3$)

Example: For $y = x^3$ (See at right)

x	-2	-2	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	-8	-1	$-\frac{1}{8}$	0	$\frac{1}{8}$	1	8

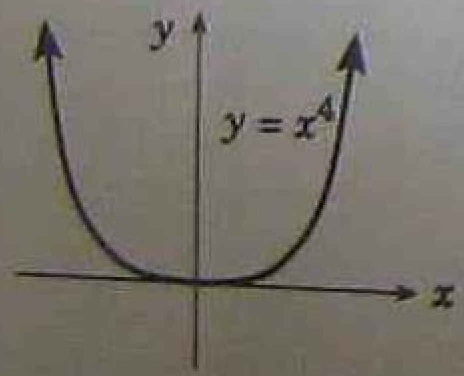
Remember: A positive number cubed gives a positive. A negative number cubed gives a negative.



5.4.2 Quartics and Quintics

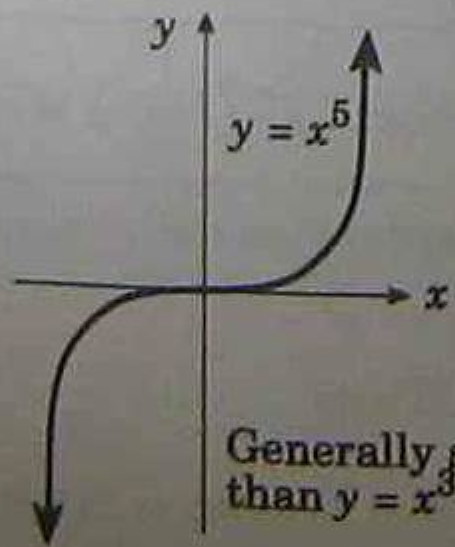
Examples

(a) **Quartic**: $y = x^4$



Generally steeper than $y = x^2$

(b) **Quintic**: $y = x^5$



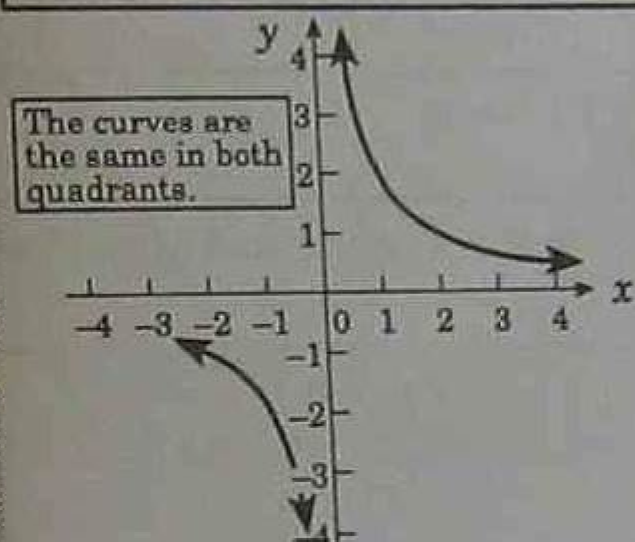
Generally steeper than $y = x^3$

5.4.3 The hyperbola $\left(y=\frac{a}{x},\ \text{or}\ xy=a\right)$

Example: Graph $y=\frac{2}{x}$

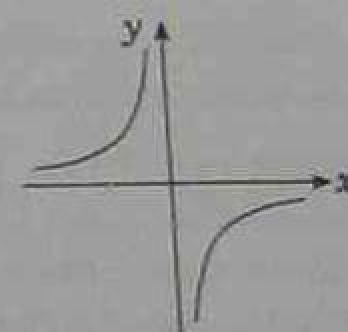
x	-4	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	4
y	$-\frac{1}{2}$	-1	-2	-4	—	4	2	1	$\frac{1}{2}$

Note that there is a problem for $x = 0$ as $y=\frac{2}{0}$ is indeterminate. Therefore the graph has a discontinuity, that is, it has two parts.



The curve gets closer and closer to the axes without touching. The axes are called the *asymptotes* of the curve.

If $a < 0$, for example, $y=-\frac{2}{x}$, we would graph it as follows:



5.4.4 The exponential (for example, $y = 2^x$)

Example: Complete the table for $y = 2^x$ and graph $y = 2^x$ on a number plane.

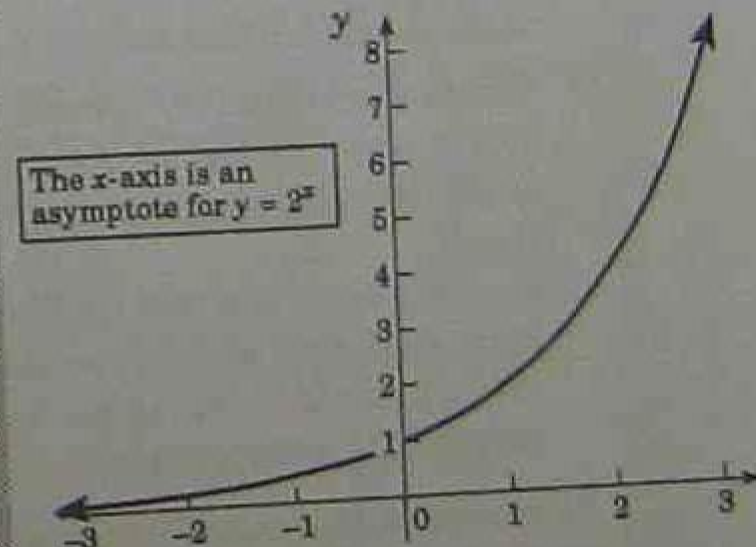
x	-3	-2	-1	0	1	2	3
y	0.125	0.25	0.5	1	2	4	8

The curve of an exponential always passes through (0, 1) and is always positive.

SOLUTION

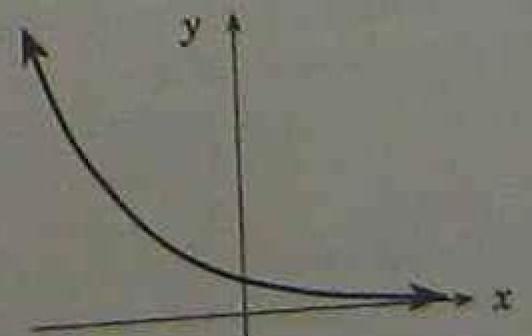
To find 2^{-3} by calculator we use:

2 $\boxed{x^y}$ 3 $\boxed{+/-}$ $\boxed{=}$

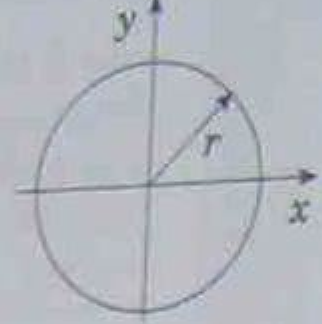


We can also graph $y = 2^{-x}$, that is:

$y = \frac{1}{2^x}$, the reciprocal of $y = 2^x$. →



5.4.5 The circle $(x^2 + y^2 = r^2)$

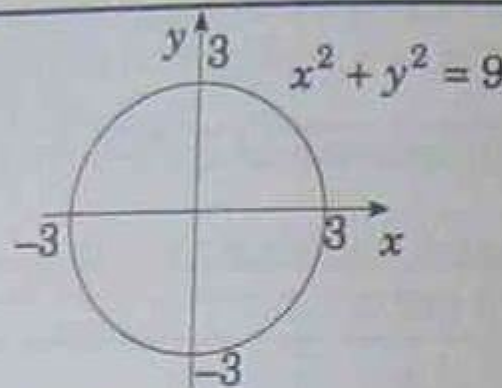


Centre: (0, 0)
Radius = r units

Example:

Graph $x^2 + y^2 = 9$

SOLUTION



Centre: (0, 0)
Radius = 3 units

5.5 Regions on the number plane

When we graph inequalities on a number plane the result is a *region* and covers half the number plane — hence the term *half-plane*.

Examples

(a) Graph $2x + y < 1$ on a number plane.

SOLUTION

- First rewrite it as $y < 1 - 2x$.
- To find the boundary of the region, let $y = 1 - 2x$ and complete a table of values:

x	0	1	2
y	1	−1	−3

- Check the inequality symbol:
  If it is $>$ or $<$ we have a broken line.
  If it is $\geq$ or $\leq$ we have an unbroken line.
  In this case it is a broken line.
- Now we decide which side of this line we shade to represent the points that satisfy the inequality.

satisfy the inequality.

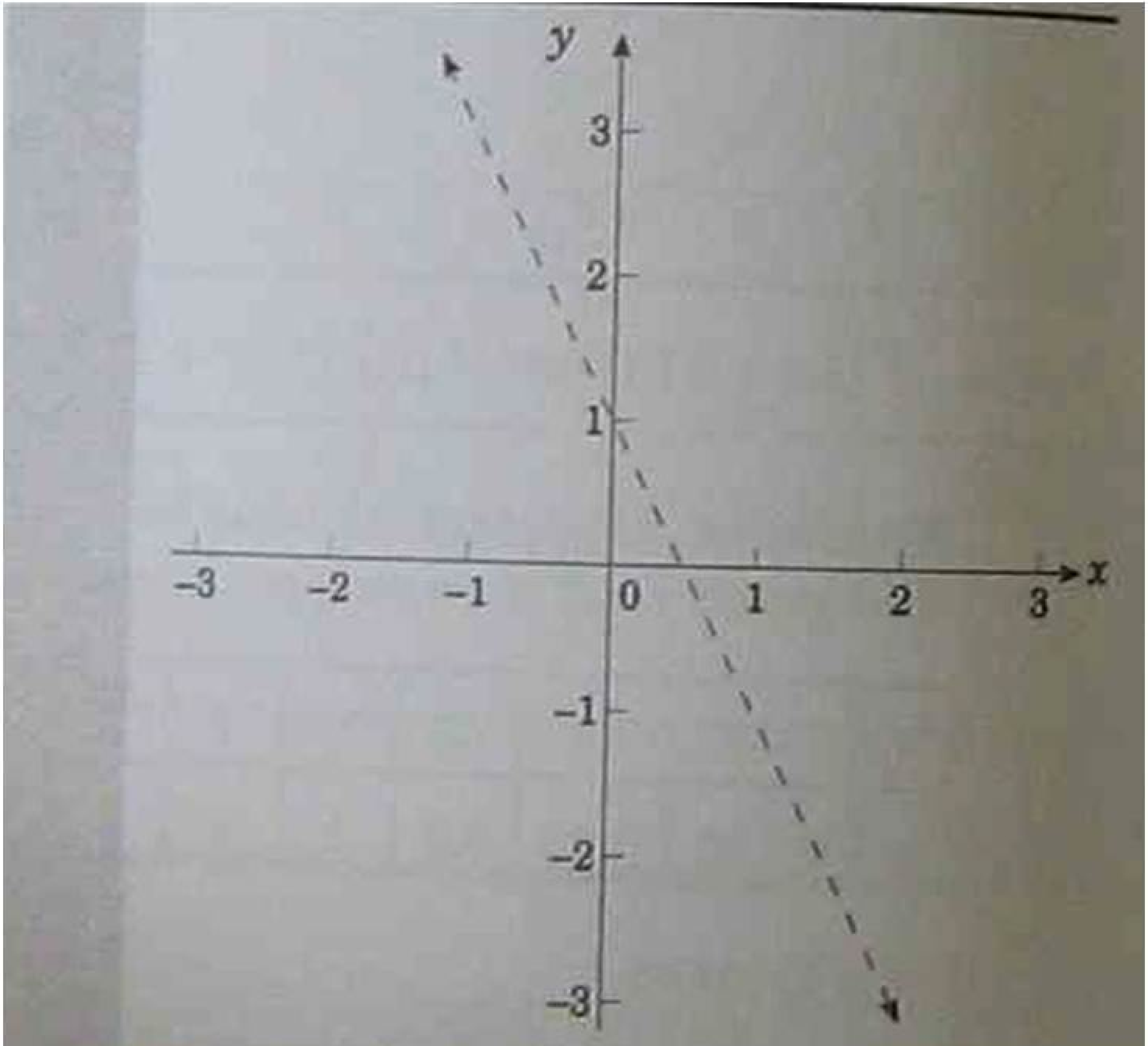
Substitute a representative point. The point (0, 0) is always easy to substitute and it is **not** on the broken line. Substitute (0, 0) in

$$y < 1 - 2x$$

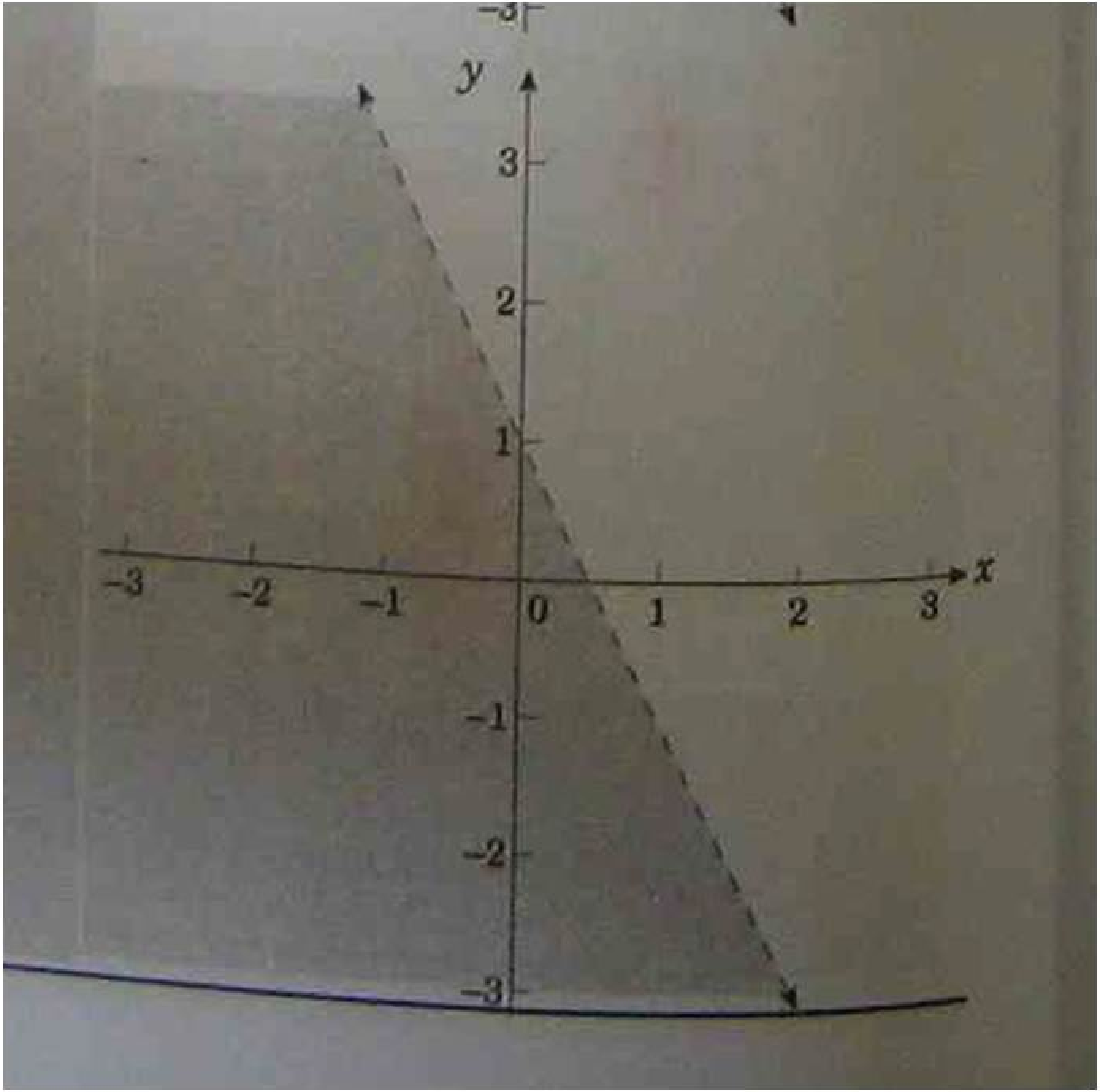
$$0 < 1 - 2(0),$$

that is, $0 < 1$? ... Yes.

Therefore all points on the side of the broken line that (0, 0) was on **must** satisfy, so we shade that side.



y
x
3
2
1
0
−1
−2
−3
−3
−2
−1
1
2
3



y
x
3
2
1
0
-1
-2
-3
-3
-2
-1
1
2
3

(b) Graph the half-plane $y \le x^2 - 2$ on a number plane.

SOLUTION

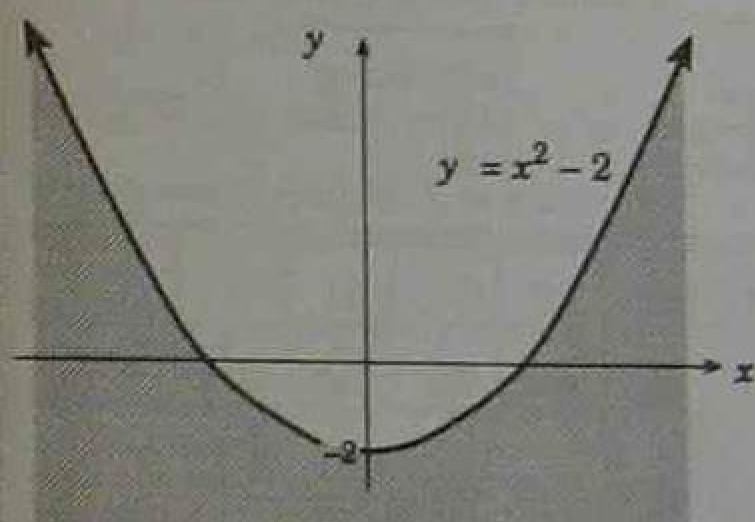
Thus graph $y = x^2 - 2$, with an unbroken line (because of $\le$).

Choose (0, 0) and substitute in $y \le x^2 - 2$,

that is, $0 \le 0^2 - 2$

$0 \le -2$? ... No!

Therefore we shade the side that (0, 0) is not on.



(c) Determine whether the points (–1, 2) or (3, -1) lie in the shaded region given as $y \le 2x - 3$

SOLUTION

Substitute each point in inequality:

$(-1, 2) \quad \therefore \quad x = -1, y = 2$ in

$$y \le 2x - 3$$

$$\therefore \quad 2 \le 2(-1) - 3$$

$$2 \le -2 - 3$$

that is, $2 \le -5$? No!

$\therefore \quad (-1, 2)$ is not in $y \le 2x - 3$

$(3, -1) \quad \therefore \quad x = 3, y = -1$ in

$$y \le 2x - 3$$

$$\therefore \quad -1 \le 2(3) - 3$$

$$-1 \le 6 - 3$$

that is, $-1 \le 3$? Yes!

$\therefore \quad (3, -1)$ lies in region $y \le 2x - 3$.

Note: The above examples could have been expressed in a slightly different way using $\{(x, y): \ldots\ldots\}$ which means 'the set of points such that ...',

for example: graph $\{(x, y): x - y < 1\}$ means graph the set of points such that $x - y < 1$.

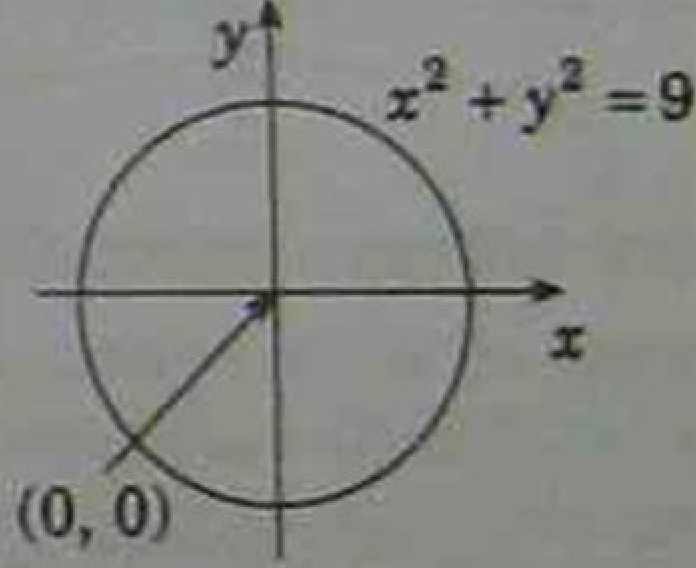
(d) Does the inequality $x^2 + y^2 < 9$ represent the inside or the outside of the circle $x^2 + y^2 = 9$?

SOLUTION

Consider a typical point inside the circle — the centre (0, 0) is an obvious choice.

Test $x = 0, y = 0$ in $x^2 + y^2 < 9$, that is, $0 + 0 < 9$.

As this is a true statement (0, 0) lies in the region $x^2 + y^2 < 9$, that is, $x^2 + y^2 < 9$ represents the inside of the circle.



If a point obviously outside was chosen, for example (4, 0), it will NOT satisfy inequation.

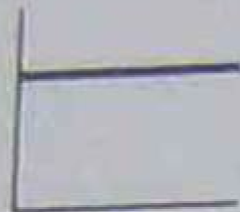
For example, $x = 4$, $y = 0$ in $x^2 + y^2 < 9$

$16 + 0 \nless 9$

(4, 0) does not lie in the region.

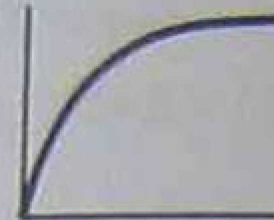
5.6 Matching graphs to physical phenomena

A line graph can represent physical phenomena as follows:



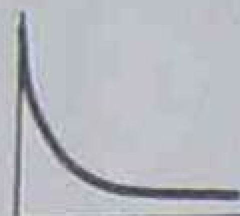
Staying the same

Increasing slowly then speeding up

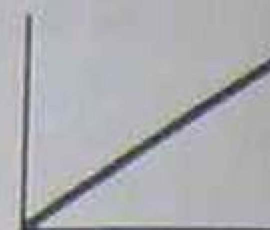


Increasing quickly then slowing down

Decreasing slowly then speeding up

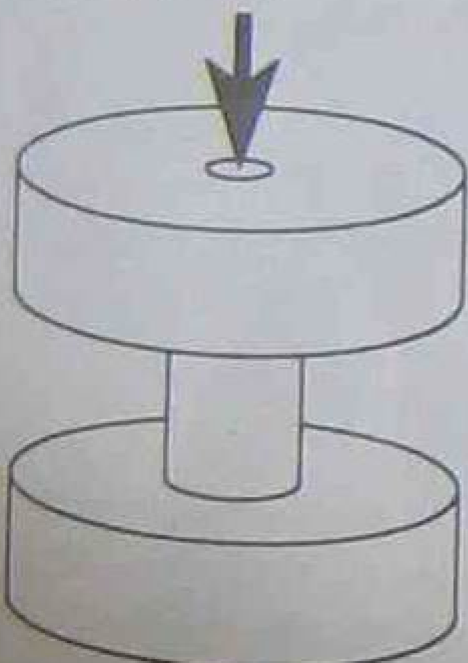


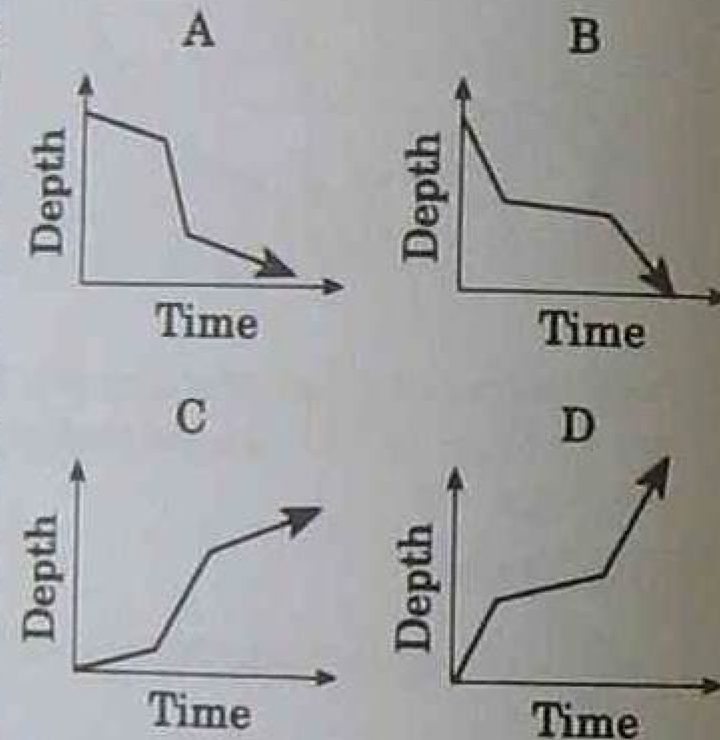
Decreasing quickly then slowing down



Increasing at the same rate

Example: Water is poured into an empty tank at a steady rate and the rise of the water level is graphed. Which graph best represents the change in water level?





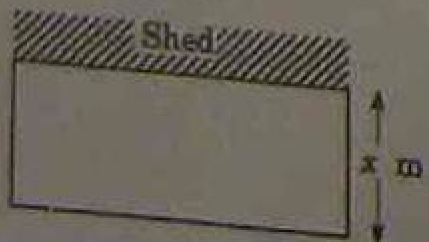
The water level would increase slowly, then quickly, and then slowly again, therefore the gradient is *gentle*, then *steep* and then *gentle*. The answer is C.

5 7 Using

5.7 Using graphs to solve problems

Examples

(a) A farmer wishes to use a 60-metre length of wire mesh to make an enclosure which will have as its fourth side an existing shed.



The width of the enclosure is x metres.

(i) Find the length of the enclosure.

SOLUTION: For the rectangle, x + length + $x = 60$, therefore, length $= 60 - 2x$.

(ii) Write down an expression for the area A enclosed.

SOLUTION

$$A = \text{length} \times \text{width}$$
$$= (60 - 2x)x$$

Therefore $A = x(60 - 2x)$.

(iii) Find the area enclosed when $x = 15$.

SOLUTION: Substitute $x = 15$ in

$$
\begin{aligned}
A &= x(60-2x)\\
&= 15(60-2(15))\\
&= 15(60-30)\\
&= 15(30)\\
&= 450
\end{aligned}
$$

The area is $450\ \mathrm{m}^2$.

(iv) Draw a graph to illustrate the possible values of the area in terms of x.

SOLUTION

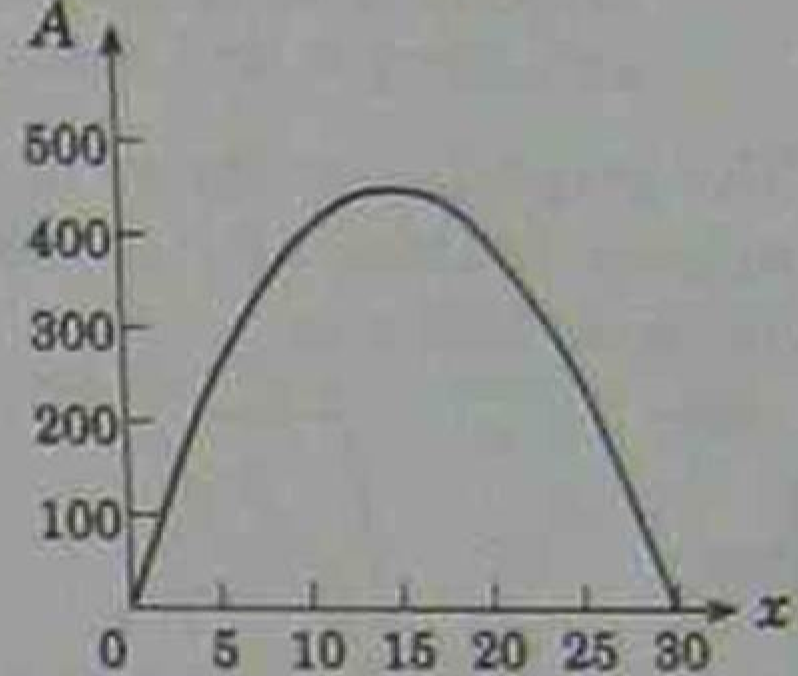
Using the equation $A = x(60 - 2x)$:

x	0	5	10	15	20	25	30
A	0	250	400	450	400	250	0

We can use a table of values or axis of symmetry, intercepts, etc.

Note: x will range between 0 and 30.

Remember that a parabola is symmetrical.



We only need the positive quadrant because width (x) and area (A) can only be positive.

(v) What is the largest area that can be enclosed?

SOLUTION: What we are asked here is: What is the *maximum value* of A?

This occurs at the maximum point on the parabola, that is, 450, therefore the largest area is 450 m^2.